Collineation Groups of the Intersection of Two Classical Unitals

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Abstract: Kestenband proved in [12] that there are only seven pairwise non-isomorphic Hermitian intersections in the desarguesian projective plane $\mathrm{PG}(2, q)$ of square order q. His classification is based on the study of the minimal polynomials of the matrices associated with the curves and leads to results of purely combinatorial nature: in fact, two Hermitian intersections from the same class might not be projectively equivalent in $\mathrm{PG}(2, q)$ and might have different collineation groups. The projective classification of Hermitian intersections in $\mathrm{PG}(2, q)$ is the main goal in this paper. It turns out that each of Kestenband's classes consists of projectively equivalent Hermitian intersections. A complete classification of the linear collineation groups preserving a Hermitian intersection is also given. © 2001 John Wiley & Sons, Inc. J Combin Designs 9: 445–459, 2001

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1. INTRODUCTION

A *unital* $\mathcal{U}$ embedded in a desarguesian projective plane $\mathrm{PG}(2, q)$ of square order q is a set of $q\sqrt{q} + 1$ points such that every line of $\mathrm{PG}(2, q)$ meets $\mathcal{U}$ in either 1 or $\sqrt{q} + 1$ points. Lines meeting $\mathcal{U}$ in $\sqrt{q} + 1$ points are called *chords* of $\mathcal{U}$. A unital $\mathcal{U}$ is *classical* if it consists of all the absolute points of a unitary polarity of $\mathrm{PG}(2, q)$ or, equivalently, of all $\mathrm{GF}(q)$-rational points of a non-singular Hermitian curve. A *Hermitian curve* $\mathcal{H}$ is the set of zeros in $\mathrm{PG}(2, q)$ of a Hermitian form. It is projectively equivalent to one of the following forms:

$R_3 : X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0$	*non-singular Hermitian curve*
$R_2 : X^{\sqrt{q}+1} - Z^{\sqrt{q}+1} = 0$	*Hermitian cone*
$R_1 : X^{\sqrt{q}+1} = 0$	*line repeated* $\sqrt{q} + 1$ *times.*

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As with unitals, a *chord* of $\mathcal{H}$ is a line of $\mathrm{PG}(2,q)$ meeting it in $\sqrt{q}+1$ rational points. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two (possibly singular) Hermitian curves in $\mathrm{PG}(2,q)$. The *Hermitian intersection* $\mathcal{E}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the set of their common $\mathrm{GF}(q)$-rational points; a *chord* of $\mathcal{E}$ is a common chord of $\mathcal{H}_1$ and $\mathcal{H}_2$. In fact, $\mathcal{E}$ and its chords are independent of the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the pencil Γ they generate over $\mathrm{GF}(\sqrt{q})$. For $i=1,2,3$, denote by Γ_i the set of curves in Γ of rank i. That is, Γ_i contains the curves of the linear system Γ that are projectively equivalent to the curve R_i. Define $n_i(\Gamma)=|\Gamma_i|$. Then,

$$n_1(\Gamma)+n_2(\Gamma)+n_3(\Gamma)=\sqrt{q}+1.$$

Kestenband [12] showed, using linear algebra techniques, that the incidence structure of the Hermitian intersection of two curves $\mathcal{H}_1$ and $\mathcal{H}_2$ belongs to one of the following classes:

1.1. Class I

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a Hermitian cone with vertex outside $\mathcal{H}_1$;
- each generator of $\mathcal{H}_2$ is a chord of $\mathcal{H}_1$;
- $|\mathcal{E}|=(\sqrt{q}+1)^2$;
- $n_3(\Gamma)=\sqrt{q}-2$; $n_2(\Gamma)=3$.

1.2. Class II

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a Hermitian cone with vertex in $\mathcal{H}_1$;
- each generator of $\mathcal{H}_2$ is a chord of $\mathcal{H}_1$;
- $|\mathcal{E}|=q+\sqrt{q}+1$;
- $n_3(\Gamma)=\sqrt{q}-1$; $n_2(\Gamma)=2$; the vertex of one of the cones in Γ_2 is outside of $\mathcal{H}_1$.

1.3. Class III

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a Hermitian cone with vertex outside $\mathcal{H}_1$;
- two generators of $\mathcal{H}_2$ are tangent to $\mathcal{H}_1$, all the others being chords;
- $|\mathcal{E}|=q+1$;
- $n_3(\Gamma)=\sqrt{q}$; $n_2(\Gamma)=1$.

1.4. Class IV

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a Hermitian cone with vertex in $\mathcal{H}_1$;
- one generator of $\mathcal{H}_2$ is tangent to $\mathcal{H}_1$, all the others being chords;
- $|\mathcal{E}|=q+1$;
- $n_3(\Gamma)=\sqrt{q}$; $n_2(\Gamma)=1$.

1.5. Class V

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a line counted $\sqrt{q}+1$ times;
- $\mathcal{H}_2$ is a chord of $\mathcal{H}_1$;
- $|\mathcal{E}| = \sqrt{q}+1$; the points of $\mathcal{E}$ form a Baer subline of $\mathrm{PG}(2,q)$;
- $n_3(\Gamma) = \sqrt{q}-1$; $n_2(\Gamma) = n_1(\Gamma) = 1$.

1.6. Class VI

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is a line counted $\sqrt{q}+1$ times;
- $\mathcal{H}_2$ is tangent to $\mathcal{H}_1$;
- $|\mathcal{E}| = 1$;
- $n_3(\Gamma) = \sqrt{q}$; $n_1(\Gamma) = 1$.

1.7. Class VII

- $\mathcal{H}_1$ is non-singular;
- $\mathcal{H}_2$ is non-singular;
- $|\mathcal{E}| = q-\sqrt{q}+1$; the points of $\mathcal{E}$ coincide with the point-orbit of a Singer subgroup of order $q-\sqrt{q}+1$;
- $n_3(\Gamma) = \sqrt{q}+1$.

A point of $\mathcal{E}$ is called *special* if it is either a vertex of a Hermitian cone in Γ, or the only common point of $\mathcal{E}$ with a generator of a Hermitian cone in Γ. There are

- no special points in classes I, V, VI and VII;
- one special point in classes II and IV;
- two special points in class III.

Our main results are the following theorems.

Theorem 1.1. *Each of the seven classes I–VII consists of pairwise projectively equivalent Hermitian intersections.*

Theorem 1.2. *The linear collineation group* $\mathrm{Aut}(\mathcal{E})$ *preserving a Hermitian intersection* $\mathcal{E}$ *acts transitively on both the special and non-special points of* $\mathcal{E}$*. The abstract structure of* $\mathrm{Aut}(\mathcal{E})$ *depends on the class containing* $\mathcal{E}$ *and is given in Theorems 2.3, 2.9, 2.12, 2.16, 2.18, 2.20, 2.22.*

2. GROUPS PRESERVING THE INTERSECTION OF TWO HERMITIAN CURVES

Let $\mathrm{Aut}(\mathcal{E})$ denote the linear collineation group preserving the Hermitian intersection $\mathcal{E}$ of two Hermitian curves $\mathcal{H}_1$ and $\mathcal{H}_2$. If Γ is the pencil generated by $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathrm{Aut}(\mathcal{E})$ induces a permutation group on the set of all Hermitian curves in Γ. More precisely, the three (possibly empty) subsets $\Gamma_i \subseteq \Gamma$, defined for $i = 1,2,3$, are invariant under the action of $\mathrm{Aut}(\mathcal{E})$. Our approach to $\mathrm{Aut}(\mathcal{E})$ is to take $\mathcal{H}_1$ from Γ_3 and $\mathcal{H}_2$ from Γ_1 or from Γ_2 when Γ_1 is empty, and determine its subgroup G consisting of all linear collineations preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$. This will be done by using several properties of the linear collineation group $\mathrm{PSU}(3, q)$ including the classification of all maximal subgroups of $\mathrm{PSU}(3, q)$, see [9,11,14]. The subgroup G turns out to be quite large and transitive on the set of all non-special points of $\mathcal{E}$. Finally, to obtain the whole $\mathrm{Aut}(\mathcal{E})$ we will also do some direct computations depending on the particular properties of $\mathcal{E}$ and the possible actions of $\mathrm{Aut}(\mathcal{E})$ on Γ.

For any $x \in \mathrm{GF}(q)$, the symbols $\mathfrak{T}[x]$ and $\mathfrak{N}[x]$ will be used to denote respectively the trace and the norm of x over the subfield $\mathrm{GF}(\sqrt{q})$; hence, $\mathfrak{T}[x] = x + x^{\sqrt{q}}$ and $\mathfrak{N}[x] = x^{\sqrt{q}+1}$.

2.1. Class I

Let $\mathcal{E}$ be a Hermitian intersection in class I. A non-singular Hermitian curve $\mathcal{H}_1$ in the pencil Γ can be assumed in its canonical form

$$\mathcal{H}_1 : X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0.$$

Since the collineation group preserving $\mathcal{H}_1$ acts transitively on the points outside $\mathcal{H}_1$, a Hermitian cone in Γ can be assumed to have its vertex V in the origin $O = (0, 0, 1)$. In particular, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are associated to a diagonal matrix, and this holds true for every curve in the pencil Γ. The three Hermitian cones in Γ are of the form

1. $\mathcal{H}_2 : \lambda X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} = 0$;
2. $\mathcal{H}_3 : (\lambda - 1)Y^{\sqrt{q}+1} + \lambda Z^{\sqrt{q}+1} = 0$;
3. $\mathcal{H}_4 : (1 - \lambda)X^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0$;

with $\lambda \in \mathrm{GF}(\sqrt{q})^{\star} \setminus \{1\}$. The properties of $\mathcal{E}$ may depend on λ. However, as Theorem 2.1 states, different choices of λ provide projectively equivalent Hermitian intersections.

Theorem 2.1. *Hermitian intersections in class I are projectively equivalent.*

Proof. Let λ, $\bar{\lambda} \in \mathrm{GF}(\sqrt{q})^{\star} \setminus \{1\}$. Then, there are elements $u, v \in \mathrm{GF}(q)^{\star}$ such that

$$u^{\sqrt{q}+1} = \frac{\lambda - 1}{\bar{\lambda} - 1}, \qquad v^{\sqrt{q}+1} = \frac{(\lambda - 1)\,\bar{\lambda}}{(\,\bar{\lambda} - 1)\lambda},$$

and let γ be the linear collineation represented by the non-singular matrix

$$\begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The projectivity γ sends $\mathcal{H}_2$ and $\mathcal{H}_3$ to the Hermitian cones of equations $\bar{\lambda}X^{\sqrt{q}+1}+Y^{\sqrt{q}+1}=0$ and $(\bar{\lambda}-1)Y^{\sqrt{q}+1}+\bar{\lambda}Z^{\sqrt{q}+1}=0$. This proves the Theorem. □

Our next aim is to determine the abstract structure and the action of $\mathrm{Aut}(\mathcal{E})$. To do this the following lemma is needed.

Lemma 2.2. *The collineation group G preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$ consists of all collineations*

$$t(\epsilon,\eta):(X,Y,Z)\rightarrow(\epsilon X,\eta Y,Z)$$

with $\epsilon^{\sqrt{q}+1}=\eta^{\sqrt{q}+1}=1$. In fact,

$$G\simeq C_{\sqrt{q}+1}\times C_{\sqrt{q}+1},$$

and it acts on the point-set of $\mathcal{E}$ as a regular permutation group.

Proof. It can be directly verified that $t(\epsilon,\eta)$ preserves both curves $\mathcal{H}_1$ and $\mathcal{H}_2$. To show the converse, let γ be a linear collineation of $\mathrm{PG}(2,q)$ preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$. Given that γ fixes the vertex $(0,0,1)$ of the Hermitian cone $\mathcal{H}_2$, the non-singular unitary matrix associated to γ is a block diagonal matrix

$$\begin{bmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{bmatrix}$$

with $b=c^{\sqrt{q}}$. Since

$$\lambda(aX+bY)^{\sqrt{q}+1}+(cX+dY)^{\sqrt{q}+1}=\mathfrak{T}[(\lambda(a^{\sqrt{q}}b)+c^{\sqrt{q}}d)X^{\sqrt{q}}Y]$$
$$+(\lambda b^{\sqrt{q}+1}+d^{\sqrt{q}+1})Y^{\sqrt{q}+1}+(\lambda a^{\sqrt{q}+1}+c^{\sqrt{q}+1})X^{\sqrt{q}+1},$$

it turns out that the condition on γ to preserve $\mathcal{H}_2$ yields

(i) $b(\lambda a^{\sqrt{q}}+d)=0$;
(ii) $(\lambda a^{\sqrt{q}+1}+b^{\sqrt{q}+1})=\lambda(\lambda b^{\sqrt{q}+1}+d^{\sqrt{q}+1})\neq 0$.

From (i), we have either $b=0$ or $b\neq 0$ and $d=-\lambda a^{\sqrt{q}}$. In the latter case,

$$b^{\sqrt{q}+1}=-\lambda a^{\sqrt{q}+1},$$

and $\lambda a^{\sqrt{q}+1}+b^{\sqrt{q}+1}=0$, against (ii). This shows that $b=0$. Then, (ii) implies $a^{\sqrt{q}+1}=d^{\sqrt{q}+1}$. Also, from $c^{\sqrt{q}+1}=b=0$ it follows that $c=0$. Hence, the collineation γ is indeed $t(a,b)$ and $G\simeq C_{\sqrt{q}+1}\times C_{\sqrt{q}+1}$. To show that G acts regularly on the points of $\mathcal{E}$, note that $\mathcal{H}_1\cap\mathcal{H}_2$ has no point on the axes. Then, the orbit of a point $P\in\mathcal{H}_1\cap\mathcal{H}_2$ under the collineation group G has size $(\sqrt{q}+1)^2$ since G has order $(\sqrt{q}+1)^2$ and no non-trivial element in G fixes a point outside the axes. On the other hand, $\mathcal{E}$ has the same size. Hence, $\mathcal{E}$ coincides with the orbit of P under G, and the claim follows. □

Theorem 2.3. *The linear collineation group* $\mathrm{Aut}(\mathcal{E})$ *preserving a Hermitian intersection in class I acts transitively on the points of* $\mathcal{E}$. *Furthermore,* $\mathrm{Aut}(\mathcal{E})$ *has order* $3(\sqrt{q}+1)^2$ *and*

$$\mathrm{Aut}(\mathcal{E}) \simeq (C_{\sqrt{q}+1} \times C_{\sqrt{q}+1}) \rtimes \mathrm{Sym}_3.$$

Proof. For any $b, c, d \in \mathrm{GF}(q)$ such that

$$b^{\sqrt{q}+1} = -\frac{1}{\lambda}, \quad c^{\sqrt{q}+1} = -\frac{(1-\lambda)^2}{\lambda}, \quad d^{\sqrt{q}+1} = \lambda(1-\lambda),$$

the group $\Sigma \cong \mathrm{Sym}_3$ generated by the linear collineations

$$\sigma_1 : (X, Y, Z) \to (Z, dX, cY),$$
$$\sigma_2 : (X, Y, Z) \to (bY, b^{-1}X, Z)$$

is a subgroup of the normaliser of G in $\mathrm{Aut}(\mathcal{E})$. In fact, $\langle G, \Sigma\rangle = G \rtimes \Sigma$ and Σ preserves Γ_2, that is the set $\{\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$. In particular, Σ is a subgroup of $\mathrm{Aut}(\mathcal{E})$. To show that $\langle G, \Sigma\rangle = \mathrm{Aut}(\mathcal{E})$, let $\tau \in \mathrm{Aut}(\mathcal{E})$. Since Σ induces the full symmetric group on Γ_2, there exists $\sigma \in \Sigma$ such that $\sigma\tau$ preserves each of the Hermitian cones $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. By virtue of the fact that the vertices of these Hermitian cones are also the vertices of the fundamental triangle, it turns out that $\sigma\tau$ is associated to a diagonal matrix

$$\begin{bmatrix} \eta & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As $\sigma\tau$ fixes $\mathcal{H}_2$, we have $\eta^{\sqrt{q}+1} = \mu^{\sqrt{q}+1}$. Also, $\mu^{\sqrt{q}+1} = 1$, because $\sigma\tau$ fixes $\mathcal{H}_3$ as well. This shows that $\sigma\tau \in G$, whence $\mathrm{Aut}(\mathcal{E}) = \langle G, \Sigma\rangle$. Now, the claim follows from the above results together with Lemma 2.2. □

2.2. Class II

Let $\mathcal{E}$ be a Hermitian intersection in class II. A non-singular Hermitian curve $\mathcal{H}_1$ in the pencil Γ is assumed in the canonical form

$$\mathcal{H}_1 : X^{\sqrt{q}+1} + YZ^{\sqrt{q}} + ZY^{\sqrt{q}} = 0,$$

while a Hermitian cone with vertex $Y_\infty = (0, 1, 0)$, say

$$\mathcal{H}_2 : \lambda X^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0,$$

is chosen to generate Γ together $\mathcal{H}_1$. One more Hermitian cone belongs to Γ, namely

$$\mathcal{H}_3 : \lambda YZ^{\sqrt{q}} + \lambda ZY^{\sqrt{q}} - Z^{\sqrt{q}+1} = 0.$$

Its vertex is the point at infinity $X_\infty = (1, 0, 0)$.

Theorem 2.4. *Hermitian intersections in class II are projectively equivalent.*

Proof. Let $\bar{\Gamma}$ be another pencil that defines a Hermitian intersection in class II. Without loss of generality, we may assume that $\bar{\Gamma}$ is generated by $\mathcal{H}_1$ together with the Hermitian cone of equation

$$\bar{\lambda}X^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0,$$

where $\bar{\lambda} \in \mathrm{GF}(\sqrt{q})^{\star}$. Arguing as in the proof of Theorem 2.1, choose an element $u \in \mathrm{GF}(q)^{\star}$ such that $u^{\sqrt{q}+1} = \lambda / \bar{\lambda}$. Let γ be the linear collineation represented by the non-singular matrix

$$\begin{bmatrix} u & 0 & 0 \\ 0 & u^{\sqrt{q}+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The projectivity γ sends $\mathcal{H}_2$ and $\mathcal{H}_3$ to the Hermitian cones of equations $\bar{\lambda}X^{\sqrt{q}+1} + Z^{\sqrt{q}+1} = 0$ and $\bar{\lambda}YZ^{\sqrt{q}} + \bar{\lambda}ZY^{\sqrt{q}} - Z^{\sqrt{q}+1} = 0$, whence the claim follows. □

We now determine the structure and the action of the linear collineation group $\mathrm{Aut}(\mathcal{E})$ preserving $\mathcal{E}$. By Theorem 2.4, we may assume $\lambda = -1$ without loss of generality. Hence, the vertex of $\mathcal{H}_3$ is the point $(0, 1, 0)$.

Lemma 2.5. *A linear collineation γ belongs to Aut$(\mathcal{E})$ if and only if*

$$\gamma(a, c, d) : (X, Y, Z) \to (aX, Y + cZ, dZ),$$

with

(i) $d \in \mathrm{GF}(\sqrt{q})^{*}$;
(ii) $\mathfrak{T}(c) = 1 - d$;
(iii) $\mathfrak{N}[a] = d^2$.

Proof. The collineation γ fixes the vertices $(1, 0, 0)$ and $(0, 1, 0)$ of both Hermitian cones in Γ. Hence, it is represented by a non-singular matrix of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & 1 & c \\ 0 & 0 & d \end{bmatrix}.$$

A necessary and sufficient condition for γ to preserve $\mathcal{H}_2$ is $b = 0$ together with $a^{\sqrt{q}+1} = d^{\sqrt{q}+1}$. A straightforward computation shows that γ preserves $\mathcal{H}_3$ if and only if and if and only if both (i) and (ii) are satisfied. □

Lemma 2.6. *Let G be the subgroup of* Aut$(\mathcal{E})$ *preserving $\mathcal{H}_1$. Then,* Aut$(\mathcal{E}) = C_{\sqrt{q}-1}G$, *where $C_{\sqrt{q}-1}$ is the cyclic group consisting of all collineations*

$$\phi(d) : (X, Y, Z) \to (X, Y, dZ)$$

with $d \in GF(\sqrt{q})^{\star}$.

Proof. A direct computation shows that the collineation $\gamma(a,c,d)$ in Lemma 2.5 preserves $\mathcal{H}_1$ provided that $d=1, \mathfrak{I}(c)=0$ and $\mathfrak{R}[a]=1$. Hence, every element in $\mathrm{Aut}(\mathcal{H})$ can be written as a the product of an element in $C_{\sqrt{q}-1}$ by an element in G. □

Lemma 2.7. *The group G has order* $(\sqrt{q}+1)\sqrt{q}$ *and it is isomorphic to the semidirect product of an elementary abelian normal subgroup of order* $\sqrt{q}$ *by a cyclic group of order* $\sqrt{q}+1$.

Proof. We verify directly that the collineations $\gamma(a,c,1)$ with $a=1$ form an elementary abelian normal subgroup $E_{\sqrt{q}}$ of order $\sqrt{q}$ while those with $c=0$ constitute a cyclic subgroup $C_{\sqrt{q}+1}$ of order $\sqrt{q}+1$. To complete the proof it suffices to check that every element in G is the product of two elements, one from the former and one from the latter subgroups. □

Lemma 2.8. *The group G acts on the points of* $\mathcal{E}$ *distinct from* $(0,1,0)$ *as a regular permutation group.*

Proof. With the notation introduced in the previous lemma, the cyclic subgroup $C_{\sqrt{q}+1}$ acts transitively on the generators of $\mathcal{H}_2$, while the elementary abelian subgroup $E_{\sqrt{q}}$ is transitive on those of $\mathcal{H}_3$ distinct from $[Z=0]$. This implies the transitivity of G on the points of $\mathcal{E}$ distinct from $(0,1,0)$. On the other hand, G has order $\sqrt{q}(\sqrt{q}+1)$ which is equal to the size of $\mathcal{E}\setminus(0,1,0)$. □

By virtue of the above lemmas, both the abstract structure and the action of $\mathrm{Aut}(\mathcal{E})$ are completely determined.

Theorem 2.9. *The linear collineation group* $\mathrm{Aut}(\mathcal{E})$ *preserving a Hermitian intersection* $\mathcal{E}$ *in class II acts transitively on the points of* $\mathcal{E}$ *distinct from the special point. Furthermore,* $\mathrm{Aut}(\mathcal{E})$ *has order* $\sqrt{q}(q-1)$*, and*

$$\mathrm{Aut}(\mathcal{E}) \simeq C_{\sqrt{q}-1} \times (E_{\sqrt{q}} \rtimes C_{\sqrt{q}+1}).$$

2.3. Class III

Let $\mathcal{E}$ be a Hermitian intersection in class III. A non-singular Hermitian curve $\mathcal{H}_1$ in the pencil Γ is assumed to be

$$\mathcal{H}_1 : XY^{\sqrt{q}} - X^{\sqrt{q}}Y + \omega Z^{\sqrt{q}+1} = 0,$$

with $\omega^{\sqrt{q}-1} = -1$. Since the collineation group preserving $\mathcal{H}_1$ is doubly transitive on the points of $\mathcal{H}_1$, the two generators of the Hermitian cone $\mathcal{H}_2$ in $\mathcal{E}$ may be assumed to be the tangents of $\mathcal{H}_1$ at the points $(0,1,0)$ and $(1,0,0)$. Then, $\mathcal{H}_2$ has equation

$$\mathcal{H}_2 : XY^{\sqrt{q}} - uYX^{\sqrt{q}} = 0.$$

with $u^{\sqrt{q}+1}=1$. Actually, $u \neq 1$. In fact, every generator of $\mathcal{H}_2$ different from the axes must be a chord of $\mathcal{H}_1$, and this occurs if and only if $u \neq 1$. Hence, in our setting, we have just $\sqrt{q}$ pairwise distinct Hermitian intersections.

Theorem 2.10. *Hermitian intersections in class III are projectively equivalent.*

Proof. For every $t \in GF(q)$ such that $u = t^{\sqrt{q}-1}$, define the linear collineation

$$\gamma(t) : (X, Y, Z) \to ((1-t)^{-1}X, Y, Z).$$

A certain amount of computation shows that $\gamma(t)$ sends $\mathcal{H}_1$ and $\mathcal{H}_2$ to $\bar{\mathcal{H}}_1$ and $\bar{\mathcal{H}}_2$ respectively, where

$$\bar{\mathcal{H}}_1 : (1-t)^{-1}XY^{\sqrt{q}} - (1-t^{\sqrt{q}})^{-1}X^{\sqrt{q}}Y + \omega Z^{\sqrt{q}+1} = 0,$$

and

$$\bar{\mathcal{H}}_2 : XY^{\sqrt{q}} - u(1-t)(1-t^{\sqrt{q}})^{-1}X^{\sqrt{q}}Y = 0.$$

Since $\mathcal{H}_1 = \bar{\mathcal{H}}_1 + t(t-1)^{-1}\bar{\mathcal{H}}_2$, it turns out that $\gamma(t)$ sends $\mathcal{E}$ to the Hermitian intersection $\bar{\mathcal{E}}$ generated by $\mathcal{H}_1$ and $\bar{\mathcal{H}}_2$. For two distinct values of t, the resulting Hermitian intersections do not coincide. In fact, if $t \in \mathrm{GF}(q)$ also satisfies the above condition, that is $\bar{t}^{\sqrt{q}-1} = u$, and $(1-t)(1-t^{\sqrt{q}})^{-1} = (1-\bar{t})(1-\bar{t})^{\sqrt{q}-1}$ holds, then we have $(1-t)(1-tu)^{-1} = (1-\bar{t})(1-u\bar{t})^{-1}$, but the latter relation implies $t = \bar{t}$. This shows that we have obtained a family of $\sqrt{q}-1$ pairwise distinct Hermitian intersections which are projectively equivalent to $\mathcal{E}$. None of them coincides with $\mathcal{E}$, as $(1-t)(1-t^{\sqrt{q}}) = 1$ implies $u = 1$ which is currently not possible. Adding $\mathcal{E}$ to that family, we obtain all possible Hermitian intersections, and this completes the proof. □

To determine the abstract structure and the action of $\mathrm{Aut}(\mathcal{E})$, we need some preliminary results.

Lemma 2.11. *The linear collineation group G preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$ consists of all collineations*

$$\gamma(a) : (X, Y, Z) \to (a^{\sqrt{q}+1}X, Y, aZ);$$
$$\delta(a) : (X, Y, Z) \to (-a^{\sqrt{q}+1}Y, X, aZ);$$

with $a \in GF(q)^*$. The subgroup $H = \{\gamma(a) \mid a \in GF(q)^*\}$ is a cyclic normal subgroup of G, and acts regularly on the points of $\mathcal{E}$ distinct from $(1,0,0)$ and $(0,1,0)$. Furthermore, if $a \in GF(\sqrt{q})^*$ and q is even, then $\delta(a)$ is an involution, and $G = \langle \delta(a) \rangle \rtimes H$; if q is odd, then $\delta(a)$ has order 4 and there exist a group C_2 such that $G = C_2 \rtimes H$.

Proof. Let g be a linear collineation preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, g preserves the fundamental triangle. More precisely, g fixes the origin $(0,0,1)$, and either interchanges $(0,1,0)$ with $(0,0,1)$, or fixes both. In the former case, g is represented by a diagonal non-singular matrix

$$M_1(a, b) = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}.$$

The collineation g preserves $\mathcal{H}_2$ if and only if $GF(\sqrt{q})^*$ contains b. For $b \in GF(\sqrt{q})^*$, the condition on g to preserve $\mathcal{H}_1$ is equivalent to $a^{\sqrt{q}+1} = b$. Hence, if g fixes the vertices of the fundamental triangle, then $g = \gamma(a)$ with a suitable element $a \in GF(\sqrt{q})^*$. A similar argument shows that if g interchanges the vertices $(1,0,0)$ and $(0,1,0)$, that is g is represented by the non-singular matrix

$$M_2(a,b) = \begin{bmatrix} 0 & b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix},$$

then the necessary and sufficient condition for g to preserve both $\mathcal{H}_1$ and $\mathcal{H}_2$ is $b = -a^{\sqrt{q}+1}$. This completes the proof of the first statement. The group H is cyclic of order $q^2 - 1$. Since $\delta(b)^{-1}\gamma(a)\delta(b) = \gamma(a^{\sqrt{q}})$ for all $a, b \in GF(q)^*$, H is a normal subgroup of G. As no non-trivial element in the subgroup H fixes a point outside the fundamental triangle, the orbit of a point $P \in \mathcal{E}$ under H has size $q^2 - 1$. Since $\mathcal{E}$ has the same size, the orbit of P and $\mathcal{E}$ coincide. This completes the proof of the second statement. Some more computation shows that $\delta(a)$ is an involution provided that $a \in \mathrm{GF}(\sqrt{q})^*$ and q is even. From this, together with the second statement, $G = \langle \delta(a) \rangle \rtimes H$ also follows. If q is odd and $a \in \mathrm{GF}(\sqrt{q})^\star$, then $\delta(a)$ has period 4, but $\delta^2 = \gamma(-1) \in G$. The result follows. □

Theorem 2.12. *The linear collineation group* Aut($\mathcal{E}$) *preserving a Hermitian intersection* $\mathcal{E}$ *in class III acts transitively on the points of* $\mathcal{E}$ *distinct from the two special points. Furthermore,* Aut($\mathcal{E}$) *has order* $2(q^2 - 1)$, *and*

$$\mathrm{Aut}(\mathcal{E}) \simeq C_2 \rtimes C_{q^2-1}.$$

Proof. We prove that every linear collineation g preserving $\mathcal{E}$ belongs to the group G introduced in the previous lemma. Actually, g preserves $\mathcal{H}_2$ and hence it suffices to prove that g also preserves $\mathcal{H}_1$. As we have already noticed in the proof of Lemma 2.11, the condition on g to preserve $\mathcal{H}_2$ implies that g is represented by one of the matrices $M_1(a,b)$, and $M_2(a,b)$, where $a \in GF(q)^*$, $b \in GF(\sqrt{q})^*$. In the former case, g sends $\mathcal{H}_1$ to the Hermitian curve $\bar{\mathcal{H}}_1$ of equation

$$XY^{\sqrt{q}} - X^{\sqrt{q}}Y + (a^{\sqrt{q}+1}b^{-1})\omega Z^{\sqrt{q}+1} = 0.$$

On the other hand, the pencil generated by $\mathcal{H}_1$ and $\mathcal{H}_2$ contains $\bar{\mathcal{H}}_2$ if and only if $a^{\sqrt{q}+1}b^{-1} = 1$. This only occurs for $b = a^{\sqrt{q}+1}$, and hence for $g \in G$. A similar argument shows that the same holds when g is represented by $M_2(a,b)$. □

2.4. Class IV

Let $\mathcal{E}$ be a Hermitian intersection in class IV. As in subsection 2.2, the non-singular Hermitian curve

$$\mathcal{H}_1 : X^{\sqrt{q}+1} + YZ^{\sqrt{q}} + Y^{\sqrt{q}}Z = 0,$$

together with the Hermitian cone with vertex $(0,1,0)$

$$\mathcal{H}_2 : ZX^{\sqrt{q}} - \lambda^2 XZ^{\sqrt{q}} = 0, \qquad \lambda^{\sqrt{q}+1} = 1$$

can be chosen to generate the pencil Γ.

Theorem 2.13. *Hermitian intersections in class IV are projectively equivalent.*

Proof. Let $\lambda, \bar{\lambda}$ be elements of GF(q) with $\lambda^{\sqrt{q}+1} = \bar{\lambda}^{\sqrt{q}+1} = 1$. Choose $a \in \mathrm{GF}(q)^\star$ such that $a^{\sqrt{q}-1} = (\lambda/\bar{\lambda}^2)$. Arguing as in the proof of Theorem 2.4, it suffices to check that the linear collineation

$$\gamma(a) : (X,Y,Z) \to (aX,Y,Z)$$

preserves $\mathcal{H}_1$ and sends $\mathcal{H}_2$ to the Hermitian cone $\bar{\mathcal{H}}_2$ of equation

$$\bar{\mathcal{H}}_2 : ZX^{\sqrt{q}} - \bar{\lambda}^2 XZ^{\sqrt{q}} = 0.$$

The result follows. □

In order to determine Aut$(\mathcal{E})$, it is possible to assume without loss of generality $\lambda = 1$.

Lemma 2.14. *The collineation group G preserving both $\mathcal{H}_1$ and $\mathcal{H}_2$ consists of all collineations*

$$t(a,c,f) : \begin{cases} X \to aX + cZ \\ Y \to -acX + a^2Y + fZ \\ Z \to Z \end{cases}$$

with $a \in GF(\sqrt{q})^\star$, $c \in GF(\sqrt{q})$ and $\mathfrak{I}[f] = -\mathfrak{R}[c]$.

Proof. A direct computation shows that every collineation $t(a,c,f)$ belongs to G. Conversely, let γ be a linear collineation preserving $\mathcal{H}_2$. Then, γ belongs to the stabilizer of the point $(0,1,0)$ and of the line $[Z=0]$. Hence, it is associated to a non-singular matrix

$$\begin{bmatrix} a & 0 & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

with $a \in \mathrm{GF}(\sqrt{q})^\star$ and $c \in \mathrm{GF}(\sqrt{q})$. Since γ preserves $\mathcal{H}_1$ as well, then $d = -ac$, $e = a^2$ and $\mathfrak{I}[f] = -\mathfrak{R}[c]$. □

Lemma 2.15. *The translation subgroup T of G consists of all collineations of the form $t(1,0,f)$. In fact, $T \simeq E_{\sqrt{q}}$, and $G/T \simeq AGL(1,\sqrt{q})$.*

Proof. The first claim follows from the previous lemma. Since $t(1,0,f)$ has order equal to the characteristic of GF$(\sqrt{q})$, and any two elements in T commute, T is an elementary abelian group and since $\mathfrak{T}(f) = 0$ has $\sqrt{q}$ solutions, T has order $\sqrt{q}$; hence, $T \simeq E_{\sqrt{q}}$. Let $\bar{G}$ be the permutation group induced by G on the set Δ of all generators of $\mathcal{H}_2$. It is easy to check that T is the kernel of the permutation representation $G \to \bar{G}$. According to Lemma 2.14, $\bar{G}$ acts on Δ as the group of all permutations $X \to aX + c$ with $a \neq 0$ and c ranging over $\mathrm{GF}(\sqrt{q})$. This completes the proof. □

The abstract structure and the action of the linear collineation group $\mathrm{Aut}(\mathcal{E})$ is given in the following theorem.

Theorem 2.16. *The linear collineation group* $\mathrm{Aut}(\mathcal{E})$ *preserving a Hermitian intersection* $\mathcal{E}$ *in class IV acts transitively on the points of* $\mathcal{E}$ *distinct from the special point. Furthermore,* $|\mathrm{Aut}(\mathcal{E})| = q(\sqrt{q} - 1)$ *and*

$$\mathrm{Aut}(\mathcal{E})/E_{\sqrt{q}} \simeq \mathrm{AGL}(1, \sqrt{q}).$$

Proof. The translation group T acts transitively on the common points of $\mathcal{E}$ and any affine line through $(0, 1, 0)$. Also, as we have seen in the proof of the previous lemma, G acts transitively on the generators of $\mathcal{H}_2$. This proves the transitivity of G on the points of $\mathcal{E}$ distinct from $(0, 1, 0)$. It remains to show that G coincides with $\mathrm{Aut}(\mathcal{E})$. Let $g \in \mathrm{Aut}(\mathcal{E})$ and $\bar{g}$ be the permutation induced by g on the set Δ of the generators of $\mathcal{H}_2$. If g is in the kernel of the permutation representation $\mathrm{Aut}(\mathcal{E}) \to \overline{\mathrm{Aut}(\mathcal{E})}$, then g is a translation. Let $T' = \langle T, g\rangle$ be the group generated by T and g. Then, T' is still a translation group. Hence no non-trivial element in T' fixes an affine point. On the other hand, T' preserves the set of all common points of $\mathcal{E}$ and an affine line through $(0, 1, 0)$. This yields that T' has order at most $\sqrt{q}$ and hence $T' = T$, that is $g \in T$. The factor group $\mathrm{Aut}(\mathcal{E})/T$ induces on Δ a permutation group containing G/T. Since $\mathrm{Aut}(\mathcal{E})/T$ preserves $\mathcal{H}_2$, it follows that $\mathrm{Aut}(\mathcal{E})/T$ consists of permutations

$$X \to aX + b,$$

with $a \in \mathrm{GF}(\sqrt{q})^{\star}$, $b \in \mathrm{GF}(\sqrt{q})$ and this shows that $G/T = \mathrm{Aut}(\mathcal{E})/T$ and therefore $G = \mathrm{Aut}(\mathcal{E})$. □

2.5. Class V

Let $\mathcal{E}$ be a Hermitian intersection in class V. The non-singular Hermitian curve

$$\mathcal{H}_1 : X^{\sqrt{q}+1} + YZ^{\sqrt{q}} + Y^{\sqrt{q}}Z = 0,$$

together with the totally degenerated Hermitian cone

$$\mathcal{H}_2 : X^{\sqrt{q}+1} = 0,$$

can be chosen to generate the pencil Γ. It turns out that every Hermitian intersection in class V is a Baer subline of $\mathrm{PG}(2, q)$. Arguing as in Section 2.4 or, alternatively, using classical results on subgeometries, see [10] Chapter 4, the following theorems can be proved.

Theorem 2.17. *Hermitian intersections in class V are projectively equivalent.*

Theorem 2.18. *The linear collineation group* Aut($\mathcal{E}$) *preserving a Hermitian intersection* $\mathcal{E}$ *in class V acts* 3*-transitively on the points of* $\mathcal{E}$ *and has order* $q^2\sqrt{q}(q-1)^2(q+1)$. *Let* $AG(2,q)$ *be the affine plane whose infinite line contains* $\mathcal{E}$ *and let* O *be a point of* $AG(2,q)$. *Then, the subgroup* K *of* Aut($\mathcal{E}$) *fixing point-wise* $\mathcal{E}$ *is the semidirect product of the full translation group* T *of* $AG(2,q)$ *by the group of all dilatations of* $AG(2,q)$ *with center* O. *Furthermore,*

$$\mathrm{Aut}(\mathcal{E})/K \simeq \mathrm{PGL}(1,\sqrt{q}).$$

2.6. Class VI

Let $\mathcal{E}$ be a Hermitian intersection in class VI. The non-singular Hermitian curve

$$\mathcal{H}_1 : X^{\sqrt{q}+1} + YZ^{\sqrt{q}} + Y^{\sqrt{q}}Z = 0,$$

together with the totally degenerated Hermitian cone

$$\mathcal{H}_2 : Z^{\sqrt{q}+1} = 0,$$

can be chosen to generate the pencil Γ. It turns out that every Hermitian intersection in class VI is reduced to a single point of $\mathrm{PG}(2,q)$. Then, the following theorems hold.

Theorem 2.19. *Hermitian intersections in class VI are projectively equivalent.*

Theorem 2.20. *The linear collineation group* Aut($\mathcal{E}$) *preserving a Hermitian intersection* $\mathcal{E}$ *in class VI has order* $q(q+1)(q-1)^2$ *and it is isomorphic to* $AGL(2,q)$.

2.7. Class VII

Let $\mathcal{E}$ be a Hermitian intersection in class VII. Then, $\mathcal{E}$ coincides with a point-orbit under a Singer subgroup of order $(q-\sqrt{q}+1)$. An interesting feature of $\mathcal{E}$ is that its points form a complete $(q-\sqrt{q}+1)$-arc in $\mathrm{PG}(2,q)$, see [2, 3, 7, 8, 13]. For an explicit presentation of the pencil Γ, we need the following results from [4], see also [5].

Let $K = \mathrm{GF}(q^3)$ be a cubic extension of $\mathrm{GF}(q)$. For any primitive (q^2+q+1)-st root of unity $a \in K$, let α be the linear collineation of $\mathrm{PG}(2,K)$ associated to the diagonal matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a^{q+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The orbit Π of the point $(1,1,1)$ under the cyclic group $A = \langle\alpha\rangle$ is a non-canonical model of $PG(2,q)$, see [6], that is, the subgeometry whose points are the points of Π and whose lines are the lines of $PG(2,K)$ meeting Π in at least 2 (and hence in $q+1$) points is isomorphic to $PG(2,q)$. The curve of equation

$$\mathcal{H}_1 : XY^{\sqrt{q}} + YZ^{\sqrt{q}} + ZX^{\sqrt{q}} = 0$$

is a Hermitian curve in the model Π of $PG(2,q)$. The collineation α sends $\mathcal{H}_1$ to the Hermitian curve of equation

$$\mathcal{H}^{\alpha} : a^{q\sqrt{q}+1}XY^{\sqrt{q}} + a^{q-\sqrt{q}+1}YZ^{\sqrt{q}} + ZX^{\sqrt{q}} = 0.$$

Since $\mathcal{H}_1 \neq \mathcal{H}^{\alpha}$, we can choose $\mathcal{H}_2$ as $\mathcal{H}^{\alpha}$. Then, $\mathcal{E}$ is represented by all the points of the form $(\epsilon, \epsilon^{\sqrt{q}}, 1)$ with $\epsilon^{q-\sqrt{q}+1} = 1$.

Theorem 2.21. *Hermitian intersections in class VII are projectively equivalent.*

Proof. This theorem is a corollary to the known result that any two Singer subgroups of the same order are conjugate under the full linear collineation group $PGL(3,q)$ of $PG(2,q)$, see [1, 9, 14]. □

Theorem 2.22. *The linear collineation group* $\mathrm{Aut}(\mathcal{E})$ *preserving a Hermitian intersection* $\mathcal{E}$ *in class VII is transitive on the points of* $\mathcal{E}$*. Furthermore,* $\mathcal{E}$ *contains a normal cyclic subgroup of order* $q - \sqrt{q} + 1$ *acting regularly on its points and*

$$\mathrm{Aut}(\mathcal{E}) = C_3 \rtimes C_{q-\sqrt{q}+1}.$$

Proof. In the above model, the Singer subgroup B of order $q - \sqrt{q} + 1$ generated by $\beta = \alpha^{q+\sqrt{q}+1}$ preserves $\mathcal{E}$ as it preserves both $\mathcal{H}_1$ and $\mathcal{H}_2$. The same holds for the linear collineation group E of order 3 generated by

$$\tau : (X, Y, Z) \to (Y, Z, X).$$

Since E normalizes B, the group $G = \langle E, B\rangle$ is the semidirect product of B by E. Hence, $G \simeq C_3 \rtimes C_{q-\sqrt{q}+1}$.

To prove that $\mathrm{Aut}(\mathcal{E}) = G$, it will be useful to regard $\mathcal{E}$ as a $(q - \sqrt{q} + 1)$-arc. Let Λ be the algebraic envelope associated to $\mathcal{E}$, viewed as an algebraic curve in the dual plane of $PG(2,q)$. Clearly, $\mathrm{Aut}(\mathcal{E})$ is an automorphism group of Λ. For q even, Λ is a projectively equivalent to a non-singular Hermitian curve $\mathcal{H}$, see [15]. The same holds for q odd, provided that projective equivalence is replaced by birational equivalence, see [4]. In any case, $\mathrm{Aut}(\mathcal{E})$ turns out to be isomorphic to a subgroup L of $PGU(3,q)$. Since $G \leq \mathrm{Aut}(\mathcal{E})$, L contains a subgroup isomorphic to G. Then, the assertion follows from the classification of all maximal subgroups of $PGU(3,q)$, see [9,11,14]. In fact, the subgroups of $PGU(3,q)$ which are the semidirect product of a cyclic group of order 3 by a cyclic group of order $q - \sqrt{q} + 1$ are all maximal in $PGU(3,q)$. □

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