A characterisation of classical unitals

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Abstract

A short proof is given for the main result of [1]:

Theorem 1. *A unital in* $\mathrm{PG}(2,q)$ *is classical if and only if it is preserved by a cyclic linear collineation group of order* $q-\sqrt{q}+1$.

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1 Introduction

A unital in a desarguesian projective plane $\mathrm{PG}(2,q)$ of square order q, is a set $\mathcal{U}$ of $q\sqrt{q}+1$ points such that any line of $\mathrm{PG}(2,q)$ meets $\mathcal{U}$ in either 1 or $\sqrt{q}+1$ points. The absolute points of an unitary polarity of $\mathrm{PG}(2,q)$ form a unital which is called the *classical* (or *Hermitian*) unital. The linear collineation group of $\mathrm{PG}(2,q)$ preserving a classical unital is $\mathrm{PGU}(3,q)$. By a theorem due to Hoffer [4] this group-theoretic property characterises classical unitals: if a unital $\mathcal{U}$ is preserved by a collineation group isomorphic to $\mathrm{PSU}(3,q)$, then $\mathcal{U}$ is classical. Cossidente, Ebert and Korchmáros [1] showed that Hoffer's result holds true under some weaker assumption, namely for unitals preserved by a Singer subgroup of $\mathrm{PGL}(3,q)$ of order $q-\sqrt{q}+1$. Their proof heavily depends on previous results concerning cyclic partitions of $\mathrm{PG}(2,q)$ in Baer sub-planes. A different and shorter proof of this result is the purpose of the present note.

2 Proof of Theorem 1

A projective plane Π is called *cyclic* if it admits a Singer group. Here a Singer group is defined to be a cyclic collineation group acting transitively on the points (and the lines) of Π.

The following useful model for $\mathrm{PG}(2,q)$, viewed as a cyclic plane, comes from [2]. Let $\mathcal{F} = \mathrm{GF}(q)$, where $q = p^h$ with h even and p an odd prime, and let $\mathcal{E} = \mathrm{GF}(q^3)$ be a cubic extension of $\mathcal{F}$. Then, the projective plane $\mathrm{PG}(2,\mathcal{F})$ embeds in a natural way into $\mathrm{PG}(2,\mathcal{E})$. Consider now the linear collineation γ of $\mathrm{PG}(2,\mathcal{E})$ given by

$$\gamma : \left\{ \begin{array}{rcl} \rho {x_1}' & = & bx \\ \rho {x_2}' & = & b^{q+1}x_2 \\ \rho {x_3}' & = & x_3, \end{array} \right.$$

where b is a primitive (q^2+q+1)-st root of unity over $\mathcal{E}$. Clearly, γ has order q^2+q+1. Furthermore, γ fixes each vertex of the fundamental triangle $A_1A_2A_3$ of $\mathrm{PG}(2,\mathcal{E})$.

The orbit of the point $E(1,1,1)$ under $G = \langle \gamma \rangle$ is given by the set

$$\Pi = \{\gamma^i(E) : i = 0, 1 \ldots, q^2+q\} = \{(c, c^{(q+1)}, 1) : c^{q^2+q+1} = 1, c \in \mathcal{E}\}.$$

Such a set Π may be endowed with the structure of the sub-geometry of $\mathrm{PG}(2,\mathcal{E})$ induced by the lines meeting Π in at least two points. In particular, the line of equation $X+Y+Z=0$ meets Π in the points of the form $(c, c^{q+1}, 1)$ with c ranging over the roots of the polynomial $X^{q+1}+X+1$. In fact, this subgeometry Π is a projective plane.

Proposition 1. *([2, Proposition 1]). The set* Π *is a cyclic projective plane which is isomorphic to* $\mathrm{PG}(2,\mathcal{F})$*. More precisely,* Π *is a projective sub-plane of* $\mathrm{PG}(2,\mathcal{E})$ *lying in a non-classical position; that is* $\Pi \neq \mathrm{PG}(2,\mathcal{F})$*. The lines of* Π *have equation* $tX + t^{q+1}Y + Z = 0$ *with* t *running over the* (q^2+q+1)*-st roots of unity, and form the line-orbit of* $x+y+1=0$ *under* G*.*

In order to simplify the notation, the symbol (i) will be used to denote the point of Π with coordinates $(b^i, b^{i(q+1)}, 1)$. Similarly, $[i]$ will indicate the line of Π of equation $b^iX + b^{i(q+1)}Y + Z = 0$.

Let $\mathcal{H}$ define the set of all points of Π lying on the algebraic plane curve of equation:

$$\mathcal{H} : XY^{\sqrt{q}} + YZ^{\sqrt{q}} + ZX^{\sqrt{q}} = 0.$$

A direct computation proves the following result.

Proposition 2. *The stabiliser of* $\mathcal{H}$ *in* G *is the subgroup* K *of order* $q - \sqrt{q} + 1$ *generated by* $\gamma^{(q+\sqrt{q}+1)}$*.*

Proposition 3. *The set* $\mathcal{H}$ *is a classical unital.*

Proof. We begin by proving that the involutory mapping

$$\varphi : \left\{ \begin{array}{rcl} (i) & \to & [iq\sqrt{q}] \\ {[i]} & \to & (iq\sqrt{q}) \end{array} \right.$$

is a non-degenerate polarity. Since Π is a cyclic plane, it is enough to show that φ sends lines through (0) to points incident with $[0]$ and vice-versa. The line $[i]$ is incident with the point (0) if and only if $b^i + b^{i(q+1)} + 1 = 0$ that is $b^{iq\sqrt{q}} + b^{i(q+1)q\sqrt{q}} + 1 = 0$. As $(iq\sqrt{q})$ is the image point of $[i]$, the first assertion follows. A similar argument proves the converse. A direct computation show also that the set of all self-conjugate points of φ coincides with $\mathcal{H}$. The classification of polarities of $\mathrm{PG}(2,q)$ implies that the polarity φ is either orthogonal or unitary. Hence, in order to get the result, it remains to prove that the former possibility cannot actually occur. It is well known that the set of all self-conjugate points of an orthogonal polarity is a (non-degenerate) conic for q odd and a line for q even. On the other hand, no collineation group of $\mathrm{PG}(2,q)$ which preserves either a conic or a line contains a cyclic subgroup of order $q-\sqrt{q}+1$. Hence, Proposition 2 rules out the possibility for φ not to be unitary. □

Since K is a normal in G, it is possible to construct a quotient incidence structure Π_0 in the following way: define *thick points* as the point-orbits of Π under K, *thick lines* as the orbits of $\mathcal{H}$ under G, and incidence as inclusion. Note that the factor group G/K is a Singer group for Π_0, as it acts regularly on the set of thick points as well as on the set of thick lines.

Proposition 4. *The incidence structure Π_0 is a projective plane of order $\sqrt{q}$.*

Proof. Since the index $[G:K]$ is equal to $q+\sqrt{q}+1$, the number of thick points in Π_0 is $q+\sqrt{q}+1$. According to Proposition 2, the subgroup of G which preserves $\mathcal{H}$ is K; hence we have $q+\sqrt{q}+1$ thick lines as well. Furthermore, every thick line is incident with $\sqrt{q}+1$ thick points, as the size of $\mathcal{H}$ is $q\sqrt{q}+1$. By [3, 3.2.3(m)], in order to prove that Π_0 is a projective plane of order $\sqrt{q}$, it now suffices to check that two thick lines share at most one thick point. The number of the common points of two distinct classical unitals in Π is at most $(\sqrt{q}+1)^2$, see [5]. Since the thick lines of Π_0 are classical unitals of Π , for $q>9$ the assertion follows from the fact that $(\sqrt{q}+1)^2 < 2(q-\sqrt{q}+1)$. For $q=4,9$ a direct counting argument proves the assertion. □

The following statement is verified in the proof of Proposition 3.2 of [1].

Proposition 5. *Every unital meets every classical unital in a non-empty set.*

We are now in position to prove Theorem 1.

Theorem 1. *A unital $\mathcal{U}$ in $\mathrm{PG}(2,q)$ is classical if and only if it is preserved by a cyclic linear collineation group of order $q-\sqrt{q}+1$.*

Proof. We may assume without loss of generality that $\mathcal{U}$ is a unital in Π which is preserved by K. This hypothesis implies that $\mathcal{U}$ is the union of $\sqrt{q}+1$ point-orbits under K; hence $\mathcal{U}$ can be viewed as a set Δ of $\sqrt{q}+1$ thick points in Π_0. In order to prove Theorem 1 it remains to show that Δ is actually a thick line. In a projective plane of order $\sqrt{q}$, the only set of $\sqrt{q}+1$ points that meets every line is a line. Hence, it is enough to check that $\mathcal{U}$ has non-empty intersection with $\mathcal{H}$ and every image of $\mathcal{H}$ under the action of G; this is a consequence of Proposition 5. □

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