

Line Polar Grassmann Codes of Orthogonal Type

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Abstract

Polar Grassmann codes of orthogonal type have been introduced in [1]. They are subcodes of the Grassmann code arising from the projective system defined by the Plücker embedding of a polar Grassmannian of orthogonal type. In the present paper we fully determine the minimum distance of line polar Grassmann Codes of orthogonal type for q odd.

Keywords: Grassmann codes, error correcting codes, line Polar Grassmannians.

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1. Introduction

Codes $\mathcal{C}_{m,k}$ arising from the Plücker embedding of the k -Grassmannians of m -dimensional vector spaces have been widely investigated since their first introduction in [10, 11]. They are a remarkable generalization of Reed–Muller codes of the first order and their monomial automorphism groups and minimum weights are well understood, see [8, 5, 6, 4].

In [1], the first two authors of the present paper introduced some new codes $\mathcal{P}_{n,k}$ arising from embeddings of orthogonal Grassmannians $\Delta_{n,k}$. These codes correspond to the projective system determined by the Plücker embedding of the Grassmannian $\Delta_{n,k}$ representing all totally singular k -spaces with respect to some non-degenerate quadratic form η defined on a vector space $V(2n+1, q)$ of dimension $2n+1$ over a finite field \mathbb{F}_q . An orthogonal Grassmann code $\mathcal{P}_{n,k}$ can be obtained from the ordinary Grassmann code $\mathcal{C}_{2n+1,k}$ by just deleting all the columns corresponding to k -spaces which are non-singular with respect to η ; it is thus a punctured version of $\mathcal{C}_{2n+1,k}$. For q odd, the dimension of $\mathcal{P}_{n,k}$ is the same as that of $\mathcal{G}_{2n+1,k}$, see [1]. The minimum distance d_{\min} of $\mathcal{P}_{n,k}$ is always bounded away from 1. Actually, it has been shown in [1] that for q odd, $d_{\min} \geq q^{k(n-k)+1} + q^{k(n-k)} - q$. By itself, this proves that the redundancy of these codes is somehow better than that of $\mathcal{C}_{2n+1,k}$.

In the present paper we prove the following theorem, fully determining all the parameters for the case of line orthogonal Grassmann codes (that is orthogonal polar Grassmann codes with $k=2$) for q odd.

Main Theorem. *For q odd, the minimum distance d_{\min} of the orthogonal Grassmann code $\mathcal{P}_{n,2}$ is*

$$d_{\min} = q^{4n-5} - q^{3n-4}.$$

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Furthermore, for $n > 2$ all words of minimum weight are projectively equivalent; for $n = 2$ there are two different classes of projectively equivalent minimum weight codewords.

Hence, we have the following.

Corollary 1.1. *For q odd, line polar Grassmann codes of orthogonal type are $[N, K, d_{\min}]$ -projective codes with*

$$N = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q^2 - 1)(q - 1)}, \quad K = \binom{2n+1}{2}, \quad d_{\min} = q^{4n-5} - q^{3n-4}.$$

1.1. Organization of the paper

In Section 2 we recall some well-known facts on projective systems and related codes, as well as the notion of polar Grassmannian of orthogonal type. In Section 3 we prove our main theorem.

2. Preliminaries

2.1. Projective systems and Grassmann codes

An $[N, K, d_{\min}]_q$ projective system $\Omega \subseteq \text{PG}(K-1, q)$ is a set of N points in $\text{PG}(K-1, q)$ such that there is a hyperplane Σ of $\text{PG}(K-1, q)$ with $\#(\Omega \setminus \Sigma) = d_{\min}$ and for any hyperplane Σ' of $\text{PG}(K-1, q)$,

$$\#(\Omega \setminus \Sigma') \geq d_{\min}.$$

Existence of $[N, K, d_{\min}]_q$ projective systems is equivalent to that of projective linear codes with the same parameters; see, for instance, [12]. Indeed, let Ω be a projective system and denote by G a matrix whose columns G_1, \dots, G_N are the coordinates of representatives of the points of Ω with respect to some fixed reference system. Then, G is the generator matrix of an $[N, K, d_{\min}]$ code over \mathbb{F}_q , say $\mathcal{C} = \mathcal{C}(\Omega)$. The code $\mathcal{C}(\Omega)$ is not, in general, uniquely determined, but it is unique up to code equivalence. We shall thus speak, with a slight abuse of language, of *the* code defined by Ω .

As any word c of $\mathcal{C}(\Omega)$ is of the form $c = mG$ for some row vector $m \in \mathbb{F}_q^K$, it is straightforward to see that the number of zeroes in c is the same as the number of points of Ω lying on the hyperplane Π_c of equation $m \cdot x = 0$, where $m \cdot x = \sum_{i=1}^K m_i x_i$ and $m = (m_i)_1^K$, $x = (x_i)_1^K$. The weight (i.e. the number of non-zero components) of c is then

$$\text{wt}(c) := |\Omega| - |\Omega \cap \Pi_c|. \quad (1)$$

Thus, the minimum distance d_{\min} of \mathcal{C} is

$$d_{\min} = |\Omega| - f_{\max}, \quad \text{where} \quad f_{\max} = \max_{\substack{\Sigma \subseteq \text{PG}(K-1, q) \\ \dim \Sigma = K-2}} |\Omega \cap \Sigma|. \quad (2)$$

We point out that any projective code $\mathcal{C}(\Omega)$ can also be regarded, equivalently, as an evaluation code over Ω of degree 1. In particular, when Ω spans the whole of $\text{PG}(K-1, q) = \text{PG}(W)$, with W the underlying vector space, then there is a bijection, induced by the standard inner product of W , between the points of the dual vector space W^* and the codewords c of $\mathcal{C}(\Omega)$.

Let $\mathcal{G}_{2n+1, k}$ be the Grassmannian of the k -subspaces of a vector space $V := V(2n+1, q)$, with $k \leq n$ and let $\eta : V \rightarrow \mathbb{F}_q$ be a non-degenerate quadratic form over V .

Denote by $\varepsilon_k : \mathcal{G}_{2n+1, k} \rightarrow \text{PG}(\wedge^k V)$ the usual Plücker embedding

$$\varepsilon_k : \text{Span}(v_1, \dots, v_k) \rightarrow \text{Span}(v_1 \wedge \dots \wedge v_k).$$

The orthogonal Grassmannian $\Delta_{n,k}$ is a geometry having as points the k -subspaces of V totally singular for η . Let $\varepsilon_k(\mathcal{G}_{2n+1,k}) := \{\varepsilon_k(X_k) : X_k \text{ is a point of } \mathcal{G}_{2n+1,k}\}$ and $\varepsilon_k(\Delta_{n,k}) = \{\varepsilon_k(\tilde{X}_k) : \tilde{X}_k \text{ is a point of } \Delta_{n,k}\}$. Clearly, we have $\varepsilon_k(\Delta_{n,k}) \subseteq \varepsilon_k(\mathcal{G}_{2n+1,k}) \subseteq \text{PG}(\wedge^k V)$. Throughout this paper we shall denote by $\mathcal{P}_{n,k}$ the code arising from the projective system $\varepsilon_k(\Delta_{n,k})$. By [3, Theorem 1.1], if $n \geq 2$ and $k \in \{1, \dots, n\}$, then $\dim \text{Span}(\varepsilon_k(\Delta_{n,k})) = \binom{2n+1}{k}$ for q odd, while $\dim \text{Span}(\varepsilon_k(\Delta_{n,k})) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$ when q is even.

We recall that for $k < n$, any line of $\Delta_{n,k}$ is also a line of $\mathcal{G}_{2n+1,k}$. For $k = n$, the lines of $\Delta_{n,n}$ are not lines of $\mathcal{G}_{2n+1,n}$; indeed, in this case $\varepsilon_n|_{\Delta_{n,n}} : \Delta_{n,n} \rightarrow \text{PG}(\wedge^n V)$ maps the lines of $\Delta_{n,n}$ onto non-singular conics of $\text{PG}(\wedge^n V)$.

The projective system identified by $\varepsilon_k(\Delta_{n,k})$ determines a code of length $N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)} - 1}{q^{i+1} - 1}$ and dimension $K = \binom{2n+1}{k}$ or $K = \binom{2n+1}{k} - \binom{2n+1}{k-2}$ according to whether q is odd or even. The following universal property provides a well-known characterization of alternating multilinear forms; see for instance [9, Theorem 14.23].

Theorem 2.1. *Let V and U be vector spaces over the same field. A map $f : V^k \rightarrow U$ is alternating k -linear if and only if there is a linear map $\bar{f} : \wedge^k V \rightarrow U$ with $f(v_1 \wedge v_2 \wedge \dots \wedge v_k) = \bar{f}(v_1, v_2, \dots, v_k)$. The map \bar{f} is uniquely determined.*

In general, the dual space $(\wedge^k V)^* \cong \wedge^k V^*$ of $\wedge^k V$ is isomorphic to the space of all k -linear alternating forms of V . For any given non-null vector $\mathbf{v} \in \wedge^{2n+1} V \cong V(1, q) \cong \mathbb{F}_q$, we have an isomorphism $\mathcal{J}_{\mathbf{v}} : \wedge^{2n+1-k} V \rightarrow (\wedge^k V)^*$ defined by $\mathcal{J}_{\mathbf{v}}(\omega)(x) = c$ for any $\omega \in \wedge^{2n+1-k} V$ and $x \in \wedge^k V$, where $c \in \mathbb{F}_q$ is such that $\omega \wedge x = c\mathbf{v}$. Clearly, as $\mathbf{v} \neq 0$ varies in $\wedge^{2n+1} V$ we obtain different isomorphisms. For the sake of simplicity, we will say that $\omega \in \wedge^{2n+1-k} V$ acts on $x \in \wedge^k V$ as $\omega \wedge x$.

For any $k = 1, \dots, 2n$ and $\varphi \in (\wedge^k V)^*, v \in \wedge^k V$ we shall use the symbol $\langle \varphi, v \rangle$ to denote the bilinear pairing

$$\left(\wedge^k V\right)^* \times \left(\wedge^k V\right) \rightarrow \mathbb{F}_q, \langle \varphi, v \rangle = \varphi(v).$$

Since the codewords of $\mathcal{P}_{n,k}$ bijectively correspond to functionals on $\wedge^k V$, we can regard a codeword as an element of $(\wedge^k V)^* \cong \wedge^k V^*$.

In this paper we are concerned with line Grassmannians, that is we assume $k = 2$.

By Theorem 2.1, we shall implicitly identify any functional $\varphi \in (\wedge^2 V)^*$ with the (necessarily degenerate) alternating bilinear form

$$\begin{cases} V \times V \rightarrow \mathbb{F}_q \\ (x, y) \rightarrow \varphi(x \wedge y). \end{cases}$$

The *radical* of φ is the set

$$\text{Rad}(\varphi) := \{v \in V : \forall w \in V, \varphi(v, w) = 0\}.$$

This is always a vector space and its codimension in V is even. As $\dim V$ is odd, $2n - 1 \geq \dim \text{Rad}(\varphi) \geq 1$ for $\varphi \neq 0$.

We point out that it has been proved in [8] that the minimum weight codewords of the line projective Grassmann code $\mathcal{C}_{2n+1,2}$, correspond to points of $\varepsilon_{2n-1}(\mathcal{G}_{2n+1,2n-1})$; these can be regarded as non-null bilinear alternating forms of V of maximum radical. Actually, non-null bilinear forms of maximum radical may yield minimum weight codewords also for Symplectic Polar Grassmann Codes, see [2].

In the case of orthogonal line Grassmannians, not all points of $\mathcal{G}_{2n+1,2n-1}$ yield codewords of $\mathcal{P}_{n,2}$ of minimum weight. However, as a consequence of the proof of our main result, we shall see that for $n > 2$ all the codewords of minimum weight of $\mathcal{P}_{n,2}$ do indeed correspond to some $(2n - 1)$ -dimensional subspaces of V , that is to say, to bilinear alternating forms of maximum radical. In the case $n = 2$, there are two classes of minimum weight codewords: one corresponding to bilinear alternating forms of maximum radical and another corresponding to certain bilinear alternating forms with radical of dimension 1.

2.2. A recursive condition

Since $\bigwedge^k V^* \cong (\bigwedge^k V)^* \cong \bigwedge^{2n+1-k} V$, for any $\varphi \in (\bigwedge^k V)^*$ there is an element $\widehat{\varphi} \in \bigwedge^{2n+1-k} V$ such that

$$\langle \varphi, x \rangle = \widehat{\varphi} \wedge x, \quad \forall x \in \bigwedge^k V.$$

Fix now $u \in V$ and $\varphi \in (\bigwedge^k V)^*$. Then, there is a unique element $\varphi_u \in \bigwedge^{k-1} V^*$ such that $\widehat{\varphi}_u = \widehat{\varphi} \wedge u \in \bigwedge^{2n+2-k} V$.

Let \mathcal{Q} be the parabolic quadric defined by the (non-degenerate) quadratic form η . For any $u \in \mathcal{Q}$, put $V_u := u^\perp / \text{Span}(u)$. Observe that as $\langle \varphi_u, u \wedge w \rangle = \widehat{\varphi} \wedge u \wedge u \wedge w = 0$ for any $u \wedge w \in \bigwedge^{k-1} V$, the functional

$$\overline{\varphi}_u : \begin{cases} \bigwedge^{k-1} V_u \rightarrow \mathbb{F}_q \\ x + (u \wedge \bigwedge^{k-2} V) \rightarrow \varphi_u(x) \end{cases}$$

with $x \in \bigwedge^{k-1} V$ and $u \wedge \bigwedge^{k-2} V := \{u \wedge y : y \in \bigwedge^{k-2} V\}$ is well defined. Furthermore, V_u is endowed with the quadratic form $\eta_u : x + \text{Span}(u) \rightarrow \eta(x)$. Clearly, $\dim V_u = 2n - 1$. It is well known that the set of all totally singular points for η_u is a parabolic quadric of rank $n - 1$ in V_u which we shall denote by $\text{Res}_{\mathcal{Q}} u$. In other words the points of $\text{Res}_{\mathcal{Q}} u$ are the lines of \mathcal{Q} through u .

We are now ready to deduce a recursive relation on the weight of codewords, in the spirit of [8].

Lemma 2.2. *Let $\varphi \in \bigwedge^k V^*$. Then,*

$$\text{wt}(\varphi) = \frac{1}{q^k - 1} \sum_{\substack{u \in \mathcal{Q} \\ \overline{\varphi}_u \neq 0}} \text{wt}(\overline{\varphi}_u).$$

Proof. Recall that

$$\begin{aligned} \text{wt}(\varphi) &= \#\{\text{Span}(v_1, \dots, v_k) : \langle \varphi, v_1 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}(v_1, \dots, v_k) \in \Delta_{n,k}\} = \\ &= \frac{1}{|\text{GL}_k(q)|} \#\{(v_1, \dots, v_k) : \langle \varphi, v_1 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}(v_1, \dots, v_k) \in \Delta_{n,k}\}, \end{aligned} \quad (3)$$

where the list (v_1, \dots, v_k) is an ordered basis of $\text{Span}(v_1, \dots, v_k) \subset \mathcal{Q}$.

For any point $u \in \mathcal{Q}$, we have $\text{Span}(u, v_2, \dots, v_k) \in \Delta_{n,k}$ if and only if $\text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_{\mathcal{Q}} u)$, where $\Delta_{n-1,k-1}(\text{Res}_{\mathcal{Q}} u)$ is the $(k - 1)$ -Grassmannian of $\text{Res}_{\mathcal{Q}} u$ and by the symbol $\text{Span}_u(v_2, \dots, v_k)$ we mean $\text{Span}(u, v_2, \dots, v_k) / \text{Span}(u)$. Furthermore, given a space $\text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_{\mathcal{Q}} u)$, any of the q^{k-1} lists $(u, v_2 + \alpha_2 u, \dots, v_k + \alpha_k u)$ is a basis for the same totally singular k -space through u , namely $\text{Span}(u, v_2, \dots, v_k)$. Conversely, given

any totally singular k -space $W \in \Delta_{n,k}$ with $u \in W$ there are $v_2, \dots, v_k \in \text{Res}_{\mathcal{Q}}u$ such that $W = \text{Span}(u, v_2, \dots, v_k)$ and $\text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)$. Let

$$\Omega_u := \{(u, v_2 + \alpha_2 u, \dots, v_k + \alpha_k u) : \langle \varphi, u \wedge v_2 \wedge \dots \wedge v_k \rangle \neq 0, \\ \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u), \alpha_2, \dots, \alpha_k \in \mathbb{F}\}.$$

Then, we have the following disjoint union

$$\{(v_1, \dots, v_k) : \langle \varphi, v_1 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}(v_1, \dots, v_k) \in \Delta_{n,k}\} = \bigcup_{u \in \mathcal{Q}} \Omega_u. \quad (4)$$

Observe that if u is not singular, then, $\Omega_u = \emptyset$, as $\text{Span}(u, v_2, \dots, v_k) \not\subseteq \mathcal{Q}$; likewise, if $\bar{\varphi}_u = 0$, then, $\langle \bar{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle = 0$ for any v_2, \dots, v_k and, consequently, $\Omega_u = \emptyset$.

The coefficients α_i , $2 \leq i \leq k$, are arbitrary in \mathbb{F} ; thus,

$$\#\Omega_u = q^{k-1} \#\{(u, v_2, \dots, v_k) : \langle \bar{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)\}.$$

Hence,

$$|\text{GL}_k(q)| \text{wt}(\varphi) = \sum_{\substack{u \in \mathcal{Q} \\ \bar{\varphi}_u \neq 0}} \#\Omega_u = \\ = q^{k-1} \sum_{\substack{u \in \mathcal{Q} \\ \bar{\varphi}_u \neq 0}} \#\{(u, v_2, \dots, v_k) : \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)\}. \quad (5)$$

Since u is fixed,

$$\#\{(u, v_2, \dots, v_k) : \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)\} = \\ \#\{(v_2, \dots, v_k) : \langle \varphi_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)\}.$$

On the other hand, by (3) and by the definition of $\bar{\varphi}_u$,

$$|\text{GL}_{k-1}(q)| \text{wt}(\bar{\varphi}_u) = \#\{(v_2, \dots, v_k) : \langle \bar{\varphi}_u, v_2 \wedge \dots \wedge v_k \rangle \neq 0, \text{Span}_u(v_2, \dots, v_k) \in \Delta_{n-1, k-1}(\text{Res}_{\mathcal{Q}}u)\};$$

thus,

$$\text{wt}(\varphi) = q^{k-1} \frac{|\text{GL}_{k-1}(q)|}{|\text{GL}_k(q)|} \sum_{\substack{u \in \mathcal{Q} \\ \bar{\varphi}_u \neq 0}} \text{wt}(\bar{\varphi}_u) = \frac{1}{q^k - 1} \sum_{\substack{u \in \mathcal{Q} \\ \bar{\varphi}_u \neq 0}} \text{wt}(\bar{\varphi}_u). \quad (6)$$

□

3. Proof of the Main Theorem

As $\dim V$ is odd, all non-degenerate quadratic forms on V are projectively equivalent. For the purposes of the present paper we can assume without loss of generality that a basis (e_1, \dots, e_{2n+1}) has been fixed such that

$$\eta(x) := \sum_{i=1}^n x_{2i-1} x_{2i} + x_{2n+1}^2. \quad (7)$$

Let $\beta(x, y) := \eta(x + y) - \eta(x) - \eta(y)$ be the bilinear form associated with η . As in Section 2.2, denote by \mathcal{Q} the set of the non-zero totally singular vectors for η . Clearly, for any k -dimensional vector subspace W of V , then $W \in \Delta_{n,k}$ if and only if $W \subseteq \mathcal{Q}$.

Henceforth we shall work under the assumption $k = 2$. Denote by φ an arbitrary alternating bilinear form defined on V and let M and S be the matrices representing respectively β and φ with respect to the basis (e_1, \dots, e_{2n+1}) of V . Write $\perp_{\mathcal{Q}}$ for the orthogonal relation induced by η and \perp_W for the (degenerate) symplectic relation induced by φ . In particular, for $v \in V$, the symbols $v^{\perp_{\mathcal{Q}}}$ and v^{\perp_W} will respectively denote the space orthogonal to v with respect to η and φ . Likewise, when X is a subspace of V , the notations $X^{\perp_{\mathcal{Q}}}$ and X^{\perp_W} will be used to denote the spaces orthogonal to X with respect to η and φ . We shall say that a subspace X is *totally singular* if $X \leq X^{\perp_{\mathcal{Q}}}$ and *totally isotropic* if $X \leq X^{\perp_W}$.

Lemma 3.1. *Let \mathcal{Q} be a parabolic quadric with equation of the form (7), and let $p \in V$, $p \neq 0$. Denote by ρ a codeword corresponding to the hyperplane $p^{\perp_{\mathcal{Q}}}$. Then,*

$$\text{wt}(\rho) = \begin{cases} q^{2n-1} & \text{if } \eta(p) = 0 \\ q^{2n-1} - q^{n-1} & \text{if } \eta(p) \text{ is a non-zero square} \\ q^{2n-1} + q^{n-1} & \text{if } \eta(p) \text{ is a non-square.} \end{cases}$$

Proof. If $\eta(p) = 0$, then $p \in \mathcal{Q}$ and $p^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$ is a cone with basis a parabolic quadric of rank $n - 1$; it has $1 + (q^{2n-1} - q)/(q - 1)$ projective points, see [7]. The value of $\text{wt}(\rho)$ now directly follows from (1).

Suppose now p to be external to \mathcal{Q} , that is $p^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$ is a hyperbolic quadric; it is immediate to see that in this case $\text{wt}(\rho) = q^{2n-1} - q^{n-1}$. Likewise, when p is internal to \mathcal{Q} , $\text{wt}(\rho) = q^{2n-1} + q^{n-1}$.

The orthogonal group $O(V)$ stabilizing the quadric \mathcal{Q} has 3 orbits on the points of V ; these correspond respectively to totally singular, external and internal points to \mathcal{Q} . By construction, all elements in the same orbit are isometric 1-dimensional quadratic spaces. In other words, the quadratic class of $\eta(p)$ is constant on each of these orbits. In particular, the point e_{2n+1} is external to \mathcal{Q} and $\eta(e_{2n+1}) = 1$ is a square. Thus we have that external points to \mathcal{Q} correspond to those p for which $\eta(p)$ is a square, $\eta(p) \neq 0$ and internal points correspond to those for which $\eta(p)$ is a non-square. \square

3.1. Some linear algebra

Lemma 3.2. *1. For any $v \in V$, $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$ if and only if v is an eigenvector of non-zero eigenvalue of $T := M^{-1}S$.*

2. The radical $\text{Rad}(\varphi)$ of φ corresponds to the eigenspace of T of eigenvalue 0.

Proof. 1. Observe that $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$ if and only if the equations $x^T M v = 0$ and $x^T S v = 0$ are equivalent for any $x \in V$. This means that there exists an element $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that $Sv = \lambda Mv$. As M is non-singular, the latter says that v is an eigenvector of non-zero eigenvalue λ for T .

2. Let v be an eigenvector of T of eigenvalue 0. Then $M^{-1}Sv = 0$, hence $Sv = 0$ and $x^T Sv = 0$ for every $x \in V$, that is $v^{\perp_W} = V$. This means $v \in \text{Rad}(\varphi)$. \square

We can now characterize the eigenspaces of T .

Lemma 3.3. *Let μ be a non-zero eigenvalue of T and V_{μ} be the corresponding eigenspace. Then,*

(1) $\forall v \in V_{\mu}$ and $r \in \text{Rad}(\varphi)$, $r \perp_{\mathcal{Q}} v$. Hence, $V_{\mu} \leq r^{\perp_{\mathcal{Q}}}$.

- (2) The eigenspace V_μ is both totally isotropic for φ and totally singular for η .
- (3) Let $\lambda, \mu \neq 0$ be two not necessarily distinct eigenvalues of T and u, v be two corresponding eigenvectors. Then, one of the following holds:
- (a) $u \perp_{\mathcal{Q}} v$ and $u \perp_W v$.
 - (b) $\mu = -\lambda$.
- (4) If λ is an eigenvalue of T then $-\lambda$ is an eigenvalue of T .

Proof. 1. Take $v \in V_\mu$. As $Tv = M^{-1}Sv = \mu v$ we also have $\mu v^T = v^T S^T M^{-T}$. So, $v^T M^T = \mu^{-1} v^T S^T$. Let $r \in \text{Rad}(\varphi)$. Then, as $S^T = -S$, $v^T M r = \mu^{-1} v^T S^T r$ and $v^T S r = 0$ for any v , we have $v^T M r = 0$, that is $r \perp_{\mathcal{Q}} v$.

2. Let $v \in V_\mu$. Then $M^{-1}Sv = \mu v$, which implies $Sv = \mu Mv$. Hence, $v^T Sv = \mu v^T Mv$. Since $v^T Sv = 0$ and $\mu \neq 0$, we also have $v^T Mv = 0$, for every $v \in V_\mu$. Thus, V_μ is totally singular for η . Since V_μ is totally singular, for any $u \in V_\mu$ we have $u^T Mv = 0$; so, $u^T Sv = \mu u^T Mv = 0$, that is V_μ is also totally isotropic.

3. Suppose that either $u \not\perp_{\mathcal{Q}} v$ or $u \not\perp_W v$. Since, by Lemma 3.2, $u^{\perp_{\mathcal{Q}}} = u^{\perp_W}$ and $v^{\perp_{\mathcal{Q}}} = v^{\perp_W}$, we have $Mu = \lambda^{-1}Su$ and $Mv = \mu^{-1}Sv$. So, $u \not\perp_{\mathcal{Q}} v$ or $u \not\perp_W v$ implies $v^T Mu \neq 0 \neq v^T Su$. Since $M^{-1}Su = \lambda u$ and $M^{-1}Sv = \mu v$, we have

$$v^T Su = v^T S(\lambda^{-1}M^{-1}Su) = \lambda^{-1}(-M^{-1}Sv)^T Su = -(\lambda^{-1}\mu)v^T Su;$$

hence, $-\lambda^{-1}\mu = 1$.

4. Let $\lambda \neq 0$ be an eigenvalue of T and x a corresponding eigenvector. Then $M^{-1}Sx = \lambda x$ if and only if $SM^{-1}Sx = \lambda Sx$, which, in turn, is equivalent to $-(M^{-1}S)^T Sx = \lambda Sx$, that is $(M^{-1}S)^T(Sx) = -\lambda Sx$. Since $\lambda \neq 0$, Sx is an eigenvector of $(M^{-1}S)^T$ of eigenvalue $-\lambda$. Clearly, $(M^{-1}S)^T$ and $M^{-1}S$ have the same eigenvalues, so $-\lambda$ is an eigenvalue of T . \square

Corollary 3.4. Let V_λ and V_μ be two eigenspaces of non-zero eigenvalues $\lambda \neq -\mu$. Then, $V_\lambda \oplus V_\mu$ is both totally singular and totally isotropic.

3.2. Minimum weight codewords

Recall that $\varphi \in \bigwedge^2 V^*$ and, for any $u \in \mathcal{Q}$, $\bar{\varphi}_u \in V^*$. In particular, φ_u either determines a hyperplane of $V_u = u^{\perp_{\mathcal{Q}}}/\text{Span}(u)$ or it is null on V_u .

Lemma 3.5. $\bar{\varphi}_u = 0$ if and only if u is an eigenvector of T .

Proof. By Lemma 3.2, u is an eigenvector of T if and only if $u^{\perp_{\mathcal{Q}}} \subseteq u^{\perp_W}$. By definition of $\perp_{\mathcal{Q}}$, for every $v \in u^{\perp_{\mathcal{Q}}} \cap \mathcal{Q}$, we have $\text{Span}(u, v) \in \Delta_{n,2}$. However, as $v \in u^{\perp_W}$, also $\langle \varphi, u \wedge v \rangle = 0$. So, $\bar{\varphi}_u(v) = 0, \forall v \in u^{\perp_{\mathcal{Q}}}$. Thus, $\bar{\varphi}_u = 0$ on $\text{Res}_{\mathcal{Q}}u$. Conversely, reading the argument backwards, we see that if $\bar{\varphi}_u = 0$ then u is eigenvector of T . \square

We remark that $\varphi_u = 0$ if and only if $u \in \ker T$ (by Lemma 3.2(2)).

Lemma 3.6. Suppose $u \in \mathcal{Q}$ not to be an eigenvector of T . Then,

$$\text{wt}(\bar{\varphi}_u) = \begin{cases} q^{2n-3} & \text{if } \eta(Tu) = 0 \\ q^{2n-3} - q^{n-2} & \text{if } \eta(Tu) \neq 0 \text{ is a square} \\ q^{2n-3} + q^{n-2} & \text{if } \eta(Tu) \text{ is a non-square} \end{cases}$$

Proof. Let $a_u := Tu$ and let $\mathcal{Q}_u := a_u^\perp \cap \mathcal{Q}$. Note that $u \in \mathcal{Q}_u \cap u^\perp$. Indeed, $u^T MTu = u^T Su = 0$. So, $\text{wt}(\overline{\varphi}_u) = \text{wt}(\overline{\varphi}_{a_u})$. The quadric $\text{Res}_{\mathcal{Q}_u} u := (\mathcal{Q}_u \cap u^\perp) / \text{Span}(u)$ is either hyperbolic, elliptic or degenerate according as a_u is external, internal or contained in \mathcal{Q} . The result now follows from Lemma 3.1. \square

Define

$$\begin{aligned} \mathfrak{A}' &:= \{u: u \in \mathcal{Q} \text{ and } u \text{ non-eigenvector of } T\}, & A' &:= \#\mathfrak{A}'; \\ \mathfrak{B} &:= \{u: u \in \mathfrak{A}' \text{ and } Tu \in \mathcal{Q}\}, & B &:= \#\mathfrak{B}; \\ \mathfrak{C} &:= \{u: u \in \mathfrak{A}' \text{ and } \eta(Tu) \text{ is a non-square}\}, & C &:= \#\mathfrak{C}. \end{aligned}$$

By definition, both \mathfrak{B} and \mathfrak{C} are subset of \mathfrak{A}' . Using (6) we can write

$$\text{wt}(\varphi) = \frac{q^{2n-3} - q^{n-2}}{q^2 - 1} A' + \frac{q^{n-2}}{q^2 - 1} B + \frac{2q^{n-2}}{q^2 - 1} C. \quad (8)$$

Put $A = q^{2n-2} - 1 - \#\{u: u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\}$; then, (8) becomes

$$\text{wt}(\varphi) = q^{4n-5} - q^{3n-4} + \frac{q^{n-2}}{q^2 - 1} ((q^{n-1} - 1)A + B + 2C). \quad (9)$$

Clearly, $B, C \geq 0$. We investigate A more closely. Let $\text{Spec}'(T)$ be the set of non-zero eigenvalues of T and let $V_\lambda = \ker(T - \lambda I)$ be the corresponding eigenspaces for $\lambda \in \text{Spec}'(T)$. By Lemma 3.3, each space V_λ is totally singular; thus

$$A = q^{2n-2} - 1 - \sum_{\lambda \in \text{Spec}'(T)} (\#V_\lambda - 1) - \#(\ker T \cap \mathcal{Q}). \quad (10)$$

Let $r \in \mathbb{N}$ be such that $\dim \text{Rad}(\varphi) = \dim \ker T = 2(n - r) + 1$, where by Theorem 2.1, we may regard φ as a bilinear alternating form.

The non-degenerate symmetric bilinear form β induces a symmetric bilinear form β^* on V^* , defined as $\beta^*(v_1^*, v_2^*) = \beta(v_1, v_2)$ where v_1^*, v_2^* are functionals determining respectively the hyperplanes v_1^\perp and v_2^\perp . In particular, the given basis (e_1, \dots, e_{2n+1}) of V , the above correspondence determines a basis (e^1, \dots, e^{2n+1}) of V^* , where e^i , as a functional, describes the hyperplane e_i^\perp for $1 \leq i \leq 2n + 1$. As before, let also $O(V)$ be the orthogonal group stabilizing \mathcal{Q} . We have the following theorem.

Theorem 3.7. *For any $\varphi \in \bigwedge^2 V^*$ exactly one of the following conditions holds:*

- (1) $r = 1$; then $\text{wt}(\varphi) \geq q^{4n-5} - q^{3n-4}$ with equality occurring if and only if φ is in the $O(V)$ -orbit of $e^1 \wedge e^{2n+1}$;
- (2) $r > 1$ and $A > 0$: in this case $\text{wt}(\varphi) > q^{4n-5} - q^{3n-4}$;
- (3) $r > 1$ and $A < 0$: in this case $r = n = 2$ and φ is in the $O(V)$ -orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$ with $\text{wt}(\varphi) = q^3 - q^2$.

Proof. If $r = 1$, then $\dim \text{Rad}(\varphi) = 2n - 1$. As $\varphi \in \bigwedge^2 V^*$ has tensor rank 1 (i.e. is *fully decomposable*), φ determines a unique 2-dimensional subspace W_φ of V^* . In particular, the subspace W_φ is endowed with the quadratic form obtained from the restriction of β^* to W_φ . There are just 5 types of 2-dimensional quadratic spaces; they correspond respectively to the forms $f_1(x, y) = 0$, $f_2(x, y) = y^2$, $f_3(x, y) = \varepsilon y^2$, $f_4(x, y) = x^2 - \varepsilon y^2$ and $f_5(x, y) = xy$, where ε is a non-square in \mathbb{F}_q and the coordinates are with respect to a given reference system of W_φ .

For each f_i , $1 \leq i \leq 5$, there are some $\varphi_i \in \bigwedge^2 V^*$ such that $\beta^*|_{W_{\varphi_i}} \cong f_i$. Examples of such φ_i inducing, respectively, f_i for $i = 1, \dots, 5$ are the following: $\varphi_1 = e^1 \wedge e^3$, $\varphi_2 = e^1 \wedge e^{2n+1}$, $\varphi_3 = e^1 \wedge (e^3 + \varepsilon e^4)$, $\varphi_4 = e^{2n+1} \wedge (e^1 - \varepsilon e^2)$ and $\varphi_5 = e^1 \wedge e^2$.

Using Witt's extension theorem we see that there always is an isometry between a given W_{φ} and any of these spaces W_{φ_i} ($1 \leq i \leq 5$) which can be extended to an element of $O(V)$. In other words any form with $r = 1$ is equivalent to one of the aforementioned five elements of $\bigwedge^2 V^*$.

A direct computation shows that the list of possible weights is as follows:

$$\begin{aligned} \text{wt}(e^1 \wedge e^2) &= \text{wt}(e^{2n+1} \wedge (e^1 - \varepsilon e^2)) = q^{4n-5} - q^{2n-3}, \\ \text{wt}(e^1 \wedge e^3) &= q^{4n-5}, \quad \text{wt}(e^1 \wedge e^{2n+1}) = q^{4n-5} - q^{3n-4}, \\ \text{wt}(e^1 \wedge (e^3 + \varepsilon e^4)) &= q^{4n-5} + q^{3n-4}. \end{aligned}$$

As an example we will explicitly compute $\text{wt}(e^1 \wedge e^2)$. The remaining cases are analogous. Since $\varphi_5 = e^1 \wedge e^2$, we have, by (3),

$$\text{wt}(\varphi_5) = \#\{(v_1, v_2) : v_1, v_2 \in \{e_1, e_2\}^{\perp \mathcal{Q}} \cap \mathcal{Q}, \beta(e_1 + v_1, e_2 + v_2) = 0\}.$$

In particular, as

$$\beta(e_1 + v_1, e_2 + v_2) = \beta(e_1, v_2) + \beta(v_1, e_2) + \beta(e_1, e_2) + \beta(v_1, v_2) = 1 + \beta(v_1, v_2)$$

we have $\beta(v_1, v_2) = -1$. Observe that $\mathcal{Q}' := \{e_1, e_2\}^{\perp} \cap \mathcal{Q}$ is a non-singular parabolic quadric $\mathcal{Q}(2n-2, q)$ of rank $n-1$; thus it contains $(q^{2n-2} - 1)$ non-zero vectors and we can choose v_1 in $(q^{2n-2} - 1)$ ways. For each projective point $p \in \mathcal{Q}'$ with $p \notin v_1^{\perp \mathcal{Q}}$ there is exactly one vector v_2 such that $v_2 \in p$ and $\beta(v_1, v_2) = -1$. The number of such points is

$$\#\mathcal{Q}' - \#(v_1^{\perp \mathcal{Q}} \cap \mathcal{Q}') = \frac{q^{2n-2} - 1}{q - 1} - \left(\frac{q^{2n-4} - 1}{q - 1} q + 1 \right) = q^{2n-3}.$$

In particular, the overall weight of $\text{wt}(\varphi_5)$ is

$$\text{wt}(\varphi_5) := q^{2n-3}(q^{2n-2} - 1) = q^{4n-5} - q^{2n-3}.$$

The case $e^1 \wedge e^{2n+1}$ will yield words of minimum weight.

Suppose now $r > 1$. Clearly,

$$\#\ker(T) \cap \mathcal{Q} \leq \#\ker(T) - 1 = q^{2n-2r+1} - 1.$$

Furthermore, if $\lambda \in \text{Spec}'(T)$ then also $-\lambda \in \text{Spec}'(T)$ by Lemma 3.3 (4). Thus, we can write $\text{Spec}'(T) = \{\lambda_1, \dots, \lambda_\ell\} \cup \{-\lambda_1, \dots, -\lambda_\ell\}$ with $\lambda_i \neq \pm \lambda_j$ if $i \neq j$. By Corollary 3.4, the space $X^+ = \bigoplus_{i=1}^{\ell} V_{\lambda_i}$ is totally singular; hence, $\dim X^+ \leq n$ and

$$\sum_{i=1}^{\ell} \#(V_{\lambda_i} \setminus \{0\}) \leq \#X^+ - 1 \leq q^n - 1;$$

likewise, considering $X^- := \bigoplus_{i=1}^{\ell} V_{-\lambda_i}$, we get $\sum_{i=1}^{\ell} \#(V_{-\lambda_i} \setminus \{0\}) \leq q^n - 1$. Thus,

$$A \geq q^{2n-2} - q^{2n-2r+1} - 2(q^n - 1). \quad (11)$$

If $A > 0$, then $\text{wt}(\varphi) > q^{4n-5} - q^{3n-4}$. We now distinguish two cases.

Suppose that $\text{Rad}(\varphi)$ contains a singular vector u ; then, by statement (1) of Lemma 3.2 $X^+ \oplus \text{Span}(u)$ would then be a totally singular subspace; thus, $\dim X^\pm \leq n - 1$ and

$$A \geq q^{2n-2} - q^{2n-2r+1} - 2(q^{n-1} - 1) > q^{n-1}(q^{n-1} - q^{n-2} - 2) \geq 0;$$

therefore, $A > 0$. By Chevalley-Warning theorem for $2(n-r)+1 \geq 3$, the set $\text{Rad}(\varphi) \cap \mathcal{Q}$ always contains a non-zero singular vector.

Suppose now that $\text{Rad}(\varphi)$ does not contain any non-zero singular vector; then $n = r$ and, consequently, $A \geq q^{2n-2} - 1 - 2(q^n - 1)$ (where we have replaced by 1 the term $q^{2(n-r)+1}$ of (11), which was an upper bound for the number of singular vectors in $\text{Rad}(\varphi)$). This latter quantity is positive unless $n = 2$.

Therefore, $A \leq 0$ and $r > 1$ can occur only for $r = n = 2$.

If $A = 0$ then

$$\#\{u: u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\} \cup \{0\} = q^2.$$

This happens only if there exists an eigenvalue $\lambda \neq 0$ such that $V_\lambda \subseteq \mathcal{Q}$ and $\dim(V_\lambda) = 2$. By Lemma 3.3(4), also $-\lambda$ is an eigenvalue, so $V_{-\lambda} \subseteq \mathcal{Q}$. Then

$$\#\{u: u \in \mathcal{Q} \text{ and } u \text{ eigenvector of } T\} \cup \{0\} > q^2, \text{ a contradiction.}$$

Hence $A < 0$ and $r > 1$. In this case $\text{Rad}(\varphi)$ would be a one dimensional subspace of V not contained in \mathcal{Q} . We claim that actually φ is in the $O(V^*)$ -orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$. As before, let $\text{Spec}'(T) = \{\lambda_1, \dots, \lambda_\ell\} \cup \{-\lambda_1, \dots, -\lambda_\ell\}$. Since X^+ is totally singular, $\dim X \leq 2$, whence $\ell \leq 2$. If $\ell = 2$, then $\dim X^+ = \dim X^- = 2$. Thus, all four eigenspaces $V_{\pm\lambda_i}$ have dimension 1 and $\sum_{\lambda \in \text{Spec}'(T)} \#(V_\lambda \setminus \{0\}) = 4(q-1)$. It follows $A \geq q^2 - 1 - 4(q-1) = (q-2)^2 - 1 \geq 0$ and we are done. Therefore, $\ell \leq 1$. If $\ell = 0$, then $A \geq q^2 - 1 > 0$. Likewise, if $\ell = 1$ and $\dim V_{\lambda_1} = \dim V_{-\lambda_1} = 1$, then $A \geq q^2 - 1 - 2(q-1) > 0$. There remain to consider only the case $\ell = 1$ and $\dim V_\lambda = \dim V_{-\lambda} = 2$. Observe first that if there were a vector $b_3 \in V_{-\lambda} \cap V_\lambda^{\perp \mathcal{Q}}$, then $V_\lambda \oplus \text{Span}(b_3)$ would be totally singular — a contradiction, as the rank of \mathcal{Q} is 2. Therefore we can choose a basis (b_1, b_2, \dots, b_5) for V such that $V_\lambda = \text{Span}(b_1, b_3)$, $V_{-\lambda} = \text{Span}(b_2, b_4)$, $\beta(b_2, b_1) = 1$, $\beta(b_3, b_2) = 0$, $\text{Rad}(\varphi) = \text{Span}(b_5)$ and $\text{Span}(b_1, b_2, b_5)^{\perp \mathcal{Q}} = \text{Span}(b_3, b_4)$. Indeed, we may assume that b_3, b_4 are a hyperbolic pair. By construction $\beta(Tb_4, b_i) = -\beta(b_4, Tb_i) = 0$ for $i = 1, 2, 4, 5$. Hence T has matrix $\text{diag}(\lambda, -\lambda, \lambda, -\lambda, 0)$ with respect to this basis, that is $\varphi = b^1 \wedge b^2 + b^3 \wedge b^4$. We now compute $\text{wt}(\varphi)$ directly, under the assumption $n = 2$ and obtain

$$\text{wt}(\varphi) = q^3 - q^2.$$

This completes the proof of the Main Theorem. \square

Corollary 3.8. *If $n > 2$ the codewords of minimum weight all lie on the orbit of $e^1 \wedge e^{2n+1}$ under the action of the orthogonal group $O(V)$. For $n = 2$ the minimum weight codewords either lie in the orbit of $e^1 \wedge e^5$ or in the orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$.*

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