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On the intersection of Hermitian surfaces

Luca Giuzzi

Abstract. We provide a description of the configuration arising from intersection of two Hermitian surfaces in PG$(3, q)$, provided that the linear system they generate contains at least a degenerate variety.

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1. Introduction

The seven possible point–line configurations arising from the intersection of two Hermitian curves and a classification of the pencils yielding each of them are given in [2]. It has been shown in [3] that these configurations are projectively unique and their full collineation group has been determined. Given two Hermitian varieties $\mathcal{H}_1$ and $\mathcal{H}_2$, their intersection $\mathcal{E}$ is exactly the base locus of the GF$(\sqrt{q})$–linear system they generate, namely

$$\Gamma(\mathcal{H}_1, \mathcal{H}_2) = \{\mathcal{H}_1 + \lambda\mathcal{H}_2 : \lambda \in \mathrm{GF}(\sqrt{q})\}.$$

In this paper we determine the size of such an intersection depending on the number of degenerate varieties in Γ and, provided that Γ contains at least a degenerate surface, describe the actual point–line configurations arising in the 3–dimensional case.

2. Intersection numbers

The set of all singular points of a Hermitian variety $\mathcal{H}$ is a subspace rad $\mathcal{H}$, the *radical* of $\mathcal{H}$. The *rank* of a Hermitian variety in PG(n, q) is the number $r = n + 1 - \dim \mathrm{rad}\, \mathcal{H}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian hypersurfaces of PG(n, q) and denote by r_i the number of varieties of rank i in the GF$(\sqrt{q})$–pencil $\Gamma = \Gamma(\mathcal{H}_1, \mathcal{H}_2)$. The list $(r_1, \dots, r_n)$ will be called the *rank sequence* of Γ.

It has been observed in [2] that the cardinality of the base locus $\mathcal{E} = \mathcal{H}_1 \cap \mathcal{H}_2$ of Γ depends only on the rank sequence of the pencil. It turns out that some of the considerations of [2] about the 2–dimensional case may be generalised to arbitrary dimension n, as it has been done in [4].

PROPOSITION 1. *The rank sequence* $(r_1, \ldots, r_n)$ *of a pencil* Γ *of Hermitian varieties in* $\mathrm{PG}(n, q)$ *satisfies the inequality*

$$\sum_{i=1}^{n}(n-i+1)r_i \leq n+1.$$

Since, see [1], the total number of points of the non–degenerate Hermitian hypersurface $\mathcal{H}$ of $\mathrm{PG}(n, q)$ is

$$\mu(n, q) = [q^{(n+1)/2} + (-1)^n][q^{n/2} - (-1)^n)]/(q-1)],$$

it is possible to formulate the following proposition.

PROPOSITION 2. *Let* $\mathcal{H}_1$, $\mathcal{H}_2$ *be two distinct non–degenerate Hermitian varieties in* $\mathrm{PG}(n, q)$*, and let* $(r_1, \ldots, r_n)$ *be the rank sequence of the pencil* $\Gamma(\mathcal{H}_1, \mathcal{H}_2)$*. Then,*

$$|\mathcal{H}_1 \cap \mathcal{H}_2| = \eta_n(\Gamma, q) = \frac{1}{\sqrt{q}(q-1)}$$
$$\left\{(1 - q^{n+1}) + \sum_{i=1}^{n} r_i[(q\sqrt{q}\mu(i-1, q) + 1)(q^{n+1-i} - 1) - (q-1)\mu(n, q)]\right\}$$
$$+\left(1 + \frac{1}{\sqrt{q}}\right)\mu(n, q).$$

In Table 1 the possible intersection numbers for any two non–degenerate Hermitian surfaces in $\mathrm{PG}(3, q)$ are outlined. All cases may actually happen.

Any non–degenerate Hermitian surface is projectively equivalent to the one of equation

$$\mathcal{U} : X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = 0.$$

In the rest of this paper, we shall usually assume $\mathcal{U} \in \Gamma$ and, in this case, just write $\Gamma(\mathcal{H}_2)$ to denote the pencil $\Gamma(\mathcal{U}, \mathcal{H}_2)$. When no ambiguity might arise, we may omit to explicitly mention also $\mathcal{H}_2$. The symbol $\mathcal{E}$ is always used to denote the base locus of the pencil being currently considered, that is the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$. Observe that it is often convenient to choose as generators of Γ some degenerate Hermitian surfaces. A Hermitian variety $\mathcal{H}_i$ shall be usually given by means of the associated Hermitian matrix H_i.

Let $\mathrm{PGU}(4, \sqrt{q})$ be the group of all linear collineations of $\mathrm{PG}(3, q)$ fixing a non–degenerate Hermitian surface in Γ, say $\mathcal{H} \neq \mathcal{H}_2$. The base locus of the pencil $\Gamma(\mathcal{H}_2)$ is clearly projectively equivalent to the base locus of $\Gamma(\mathcal{H}_2^\sigma)$ for any $\sigma \in \mathrm{PGU}(4, \sqrt{q})$. It follows that if H_2 is the matrix associated to $\mathcal{H}_2$, then any other matrix of the form

$$H_2' = \overline{G}^t H_2 G,$$

with $G \in \mathrm{PGU}(4, \sqrt{q})$, gives a surface yielding an intersection configuration $\mathcal{E}^G$ equivalent to that determined by $\mathcal{H}_1$ and $\mathcal{H}_2$.

r_1	r_2	r_3	$\eta_3(\Gamma, q)$
0	0	0	$(q+1)^2$
0	0	1	$(q+\sqrt{q}+1)(q-\sqrt{q}+1)$
0	0	2	(q^2+1)
0	0	3	q^2-q+1
0	0	4	$(q-1)^2$
0	1	0	$q^2+q\sqrt{q}+q+1$
0	1	1	$q^2+q\sqrt{q}+1$
0	1	2	$(\sqrt{q}+1)(q\sqrt{q}-q+1)$
0	2	0	$(\sqrt{q}+1)(q\sqrt{q}+q-\sqrt{q}+1)$
1	0	0	$q\sqrt{q}+q+1$
1	0	1	$q\sqrt{q}+1$

Table 1 Possible intersection numbers for Hermitian surfaces: non–degenerate pencils

3. Description of the configurations

3.1. *Pencils with a degenerate surface of rank* 1

The simplest case to consider is when the linear system Γ contains a degenerate surface $\mathcal{C}$ of rank 1, that is a plane repeated $(q+1)$ times.

In this case, the intersection is either a degenerate or non–degenerate Hermitian curve, according as $\mathcal{C}$ is tangent or secant to all the other surfaces in the pencil. It follows from Table 1 that the first case occurs when all the surfaces in $\Gamma \setminus \{\mathcal{C}\}$ are non–degenerate, whereas the latter happens if and only if Γ contains also a surface of rank 3, a so called *Hermitian cone*.

In order to generate a configuration of the former type, consider the pencil $\Gamma(\mathcal{H}_1, \mathcal{H}_2)$ containing the surfaces given by

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A configuration of the latter type may be obtained form

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.2. *Pencils whose degenerate surfaces have all rank* 2

A Hermitian surface $\mathcal{P}$ of rank 2 is a set of $\sqrt{q}+1$ planes through a line which is the radical of $\mathcal{P}$.

PROPOSITION 3. *Suppose that* Γ *contains exactly one degenerate surface* $\mathcal{P}$ *of rank* 2*. Then, either* 1, 2 *or* $\sqrt{q}+1$ *components of* $\mathcal{P}$ *are degenerate Hermitian curves.*

Proof. The radical of $\mathcal{P}$ meets $\mathcal{E}$ in either 1, $\sqrt{q}+1$ or $q+1$ points. Let $n = |\mathrm{rad}\,\mathcal{P} \cap \mathcal{E}|$ and denote by v the number of components of $\mathcal{P}$ which meet $\mathcal{E}$ in a degenerate Hermitian curve.

(1) $n = 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v(q\sqrt{q} + q) + (\sqrt{q} + 1 - v)q\sqrt{q} + 1;$$

hence, $v = 1$.

(2) $n = \sqrt{q} + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v(q\sqrt{q} + q - \sqrt{q}) + \sqrt{q}(\sqrt{q} + 1 - v)(q - 1) + \sqrt{q} + 1;$$

hence, $v = 2$.

(3) $n = q + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v(q\sqrt{q}) + q(\sqrt{q} + 1 - v)(\sqrt{q} - 1) + q + 1;$$

hence, $v = \sqrt{q} + 1$.

□

A basis for each of the three pencils described in Proposition 3. is given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha^{\sqrt{q}} & 0 & \beta^{\sqrt{q}} & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\alpha^{\sqrt{q}+1}, \beta^{\sqrt{q}+1} \neq 0, 1$ and $\alpha^{\sqrt{q}+1} + \beta^{\sqrt{q}+1} = 0$ for case (1);

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha^{\sqrt{q}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\alpha^{\sqrt{q}+1} \neq 1$ for case (2);

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for case (3).

In general, if $\mathcal{P}$ and $\mathcal{P}'$ are any two degenerate surfaces in Γ, then $\mathcal{R} = \operatorname{rad}\mathcal{P} \cap \operatorname{rad}\mathcal{P}' = \emptyset$; otherwise, any point $V \in \mathcal{R}$ would be singular for all the surfaces in Γ, contradicting the assumption that there is at least one non–degenerate surface in the pencil.

Assume now that there are two distinct Hermitian surfaces $\mathcal{P}$ and $\mathcal{P}'$ both of rank 2 in Γ; by the previous remark, $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ have to be mutually skew lines. Furthermore, both $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ meet any non–degenerate surface of Γ in $(\sqrt{q}+1)$ points. Thus, we formulate the following proposition.

PROPOSITION 4. *The intersection of two non degenerate Hermitian surfaces $\mathcal{H}_1$, $\mathcal{H}_2$ spawning a pencil with $r_2 = 2$ is the union of all generators of $\mathcal{H}_1$ which pass through two mutually skew $(\sqrt{q}+1)$–secants.*

A pencil of this kind may be obtained from

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.3. *Pencils whose degenerate surfaces have rank* 2 *and* 3

Given a Hermitian cone $\mathcal{C}$ of vertex V and a non–degenerate Hermitian variety $\mathcal{H}$, denote by π the polar plane of V with respect to $\mathcal{H}$. Let $\Gamma'(\mathcal{C}, \mathcal{H})$ be the $\mathrm{GF}(\sqrt{q})$–linear system of Hermitian curves generated by $\mathcal{C}' = \mathcal{C} \cap \pi$ and $\mathcal{H}' = \mathcal{H} \cap \pi$.

LEMMA 5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct Hermitian cones of vertices respectively V_1 and V_2. Assume that the pencil $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ contains at least a non–singular surface and that $V_1 \notin \mathcal{E}$. Then, V_2 belongs to the polar plane of V_1 with respect to any non–degenerate Hermitian surface in Γ.*

Proof. Fix a non–degenerate Hermitian surface $\mathcal{H} \in \Gamma$ and let π be the polar plane of V_1 with respect to $\mathcal{H}$. Since $V_1 \notin \mathcal{H}$, the plane π cuts a non–singular Hermitian curve on $\mathcal{H}$. Suppose $V_2 \notin \pi$; then the line V_1V_2 would meet $\mathcal{H}$ in $\sqrt{q}+1$ points. On the other hand,

V_1V_2 meets $\mathcal{C}_1$ and $\mathcal{C}_2$ in either 1 or $q+1$ points, a contradiction. It follows that $V_2 \in \pi$ and $|V_1V_2 \cap \mathcal{E}| \leq 1$. $\square$

LEMMA 6. *Let $\mathcal{C}$ be a Hermitian cone of vertex V and $\mathcal{H}$ be a non–degenerate Hermitian surface; denote by π be the polar plane of V with respect to $\mathcal{H}$ and let also, as usual, $\Gamma = \Gamma(\mathcal{C}, \mathcal{H})$. Then,*

(1) *if $V \notin \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 + q\sqrt{q} + \sqrt{q} + 1 - \eta_2(\Gamma', q)\sqrt{q};$$

(2) *if $V \in \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 - q + |\pi \cap \mathcal{E}|.$$

Proof. Let $\mathcal{H}' = \mathcal{H} \cap \pi$ and $\mathcal{C}' = \mathcal{C} \cap \pi$. Any line through V and tangent to $\mathcal{H}$ is of the form PV with $P \in \mathcal{H}'$. If $V \notin \mathcal{H}$, then every line through V meets $\mathcal{H}$ in either 1 or $\sqrt{q}+1$ points; on the other hand, there are exactly h generators of the cone $\mathcal{C}$ tangent to $\mathcal{H}$, whence

$$\eta_3(\Gamma, q) = \eta_2(\Gamma', q) + (q\sqrt{q} + 1 - \eta_2(\Gamma', q))(\sqrt{q} + 1).$$

Assume now $V \in \mathcal{H}$. Hence, π is the tangent plane to $\mathcal{H}$ at V and $\mathcal{H}'$ consists of $\sqrt{q}+1$ lines through V. However, $\mathcal{C}'$ in this case consists of either 1 line or a degenerate Hermitian curve. In the former case we would have $\eta_3(\Gamma, q) = q^2 + q + 1$, which is not possible. Hence, $\mathcal{C}'$ is the union of $\sqrt{q}+1$ lines through V and

$$\eta_3(\Gamma, q) = \sqrt{q}(q\sqrt{q} - \sqrt{q}) + |\pi \cap \mathcal{E}|.$$

Observe that, under this assumption, all the curves in the linear system Γ' are degenerate. $\square$

Using the cardinality formula of Proposition 2 together with Lemma 6 it is possible to reconstruct the rank sequence of $\Gamma(\mathcal{C}, \mathcal{H})$ from just the rank sequence of Γ'.

LEMMA 7. *Assume Γ to contain at least one cone $\mathcal{C}$ of vertex $V \notin \mathcal{E}$, and let the rank sequence of Γ' be (r_1', r_2'). Then, the rank sequence of Γ is $(0, r_1', r_2' + 1)$.*

The plane configuration $\mathcal{E}' = \mathcal{E} \cap \pi$ clearly belongs to one of the seven classes of [2]; in order to identify those configurations we shall use the notation of that paper.

PROPOSITION 8. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma) = r_3(\Gamma) = 1$. Then, $\mathcal{E}$ is either the union of $\sqrt{q}+1$ non–degenerate Hermitian curves all with a point in common, or the union of $\sqrt{q}$ non–degenerate Hermitian curves and a degenerate Hermitian curve, all sharing a $(\sqrt{q}+1)$–secant.*

Proof. Let $\mathcal{P}$ and $\mathcal{C}$ be the only surface of rank 2 and the only Hermitian cone in Γ. Let also $L = \operatorname{rad}\mathcal{P}$ and V be the vertex of $\mathcal{C}$. Observe that $l = |L \cap \mathcal{E}| \in \{1, \sqrt{q}+1\}$.

(1) $l = 1$. Let M be the point of intersection of $\mathcal{C}$ and L; clearly $M \neq V$. If $V \in \mathcal{E}$, then there is a component π of $\mathcal{P}$ such that $V \in \pi$. However, in this case $\mathcal{C} \cap \pi = PM$, but this can not be a plane section of a non–degenerate Hermitian surface; hence, $V \notin \mathcal{E}$ follows. This being the case, all the $\sqrt{q}+1$ sections cut on $\mathcal{C}$ by $\mathcal{P}$ are non–degenerate Hermitian curves having the point M in common.

(2) $l = \sqrt{q}+1$. Let v be the number of the components of $\mathcal{P}$ which meet $\mathcal{C}$ in a degenerate Hermitian curve. Observe that $v \leq 1$ and equality occurs if and only if $V \in \mathcal{E}$. Since,

$$(q^2 + q\sqrt{q} + 1) = (\sqrt{q}+1) + (\sqrt{q}+1-v)(q\sqrt{q} - \sqrt{q}) + v(q\sqrt{q} + q^2 - \sqrt{q}),$$

we get $v = 1$ and $V \in \mathcal{E}$. □

PROPOSITION 9. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma) = 1$ and $r_3(\Gamma) = 2$. Then, $\mathcal{E}$ is the union of $\sqrt{q}+1$ non–degenerate Hermitian curves, all with a $(\sqrt{q}+1)$–secant in common.*

Proof. Let $\mathcal{P}$ be the only Hermitian surface or rank 2 in Γ and take $\mathcal{C}_1$, $\mathcal{C}_2$ as the two Hermitian cones in the pencil. As before, denote the vertices of $\mathcal{C}_1$ and $\mathcal{C}_2$ by V_1 and V_2. Put also $L = \operatorname{rad}\mathcal{P}$ and $l = |L \cap \mathcal{E}|$. Either $l = 1$ or $l = \sqrt{q}+1$. If it were $l = 1$, then $\mathcal{E}$ would be the union of $\sqrt{q}+1$ non–degenerate Hermitian curves, all with a point in common. However, this is a contradiction because of Proposition 2. Assume then $l = \sqrt{q}+1$ and denote by v the number of components of $\mathcal{P}$ meeting $\mathcal{E}$ in a degenerate Hermitian curve. Then,

$$\begin{aligned}&(\sqrt{q}+1)(q\sqrt{q} - q + 1)\\ &\quad = (\sqrt{q}+1) + (\sqrt{q}+1-v)(q\sqrt{q} - \sqrt{q}) + v(q\sqrt{q} + q^2 - \sqrt{q}).\end{aligned}$$

This is possible only if $v = 0$; that is, $\mathcal{E}$ is the union of $\sqrt{q}+1$ non–degenerate Hermitian curves, all with a $(\sqrt{q}+1)$–secant in common. □

The configuration $\mathcal{E}'$ associated with the pencil of Proposition 9 is of class (c). In particular, by [2, Lemma 5], the pencil Γ' is generated by the curve associated to the identity matrix and by a curve associated with a Hermitian matrix of minimal polynomial $m(x) = (x-\alpha)(x-\beta)$ with $\alpha \neq \beta$. Likewise, a Hermitian pencil of rank sequence (0, 1, 1) is associated with the plane configuration $\mathcal{E}'$ consisting of just one point; hence, the pencil Γ' may be generated by the curve associated with the identity matrix and by that associated with any Hermitian matrix of minimal polynomial $m(x) = (x-\lambda)^2$.

3.4. *Pencils whose degenerate surfaces have all rank* 3

All the pencils considered in this section contain at least a cone $\mathcal{C}$. We shall denote by s_1, s_2 and s_3 the number of generators of $\mathcal{C}$ meeting $\mathcal{E}$ in respectively $q+1$, $\sqrt{q}+1$ or 1 points.

PROPOSITION 10. *Assume that the pencil Γ contains exactly two cones $\mathcal{C}_1$, $\mathcal{C}_2$ of respectively vertices V_1 and V_2. Then, one of the following possibilities holds:*

(1) *both $V_1, V_2 \in \mathcal{E}$; then, $\mathcal{E}$ contains the line V_1V_2; $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(2) *$V_1 \in \mathcal{E}$, while $V_2 \notin \mathcal{E}$; $\mathcal{E}$ does not contain any line; $q\sqrt{q}$ components of $\mathcal{C}_1$ and $q(\sqrt{q}-1)$ components of $\mathcal{C}_2$ meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(3) *$V_1, V_2 \notin \mathcal{E}$; then, $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $(\sqrt{q}+1)$ points.*

Proof. Let V be the vertex of any cone $\mathcal{C}$ in Γ. Observe that for $V \notin \mathcal{E}$,

$$q^2+1 = s_2(\sqrt{q}+1) + (q\sqrt{q}+1-s_2);$$

hence, $s_2 = q\sqrt{q}-q$. On the other hand, if $V \in \mathcal{E}$

$$q^2+1 = s_1q + s_2\sqrt{q}+1;$$

hence, $s_2 = \sqrt{q}(q-s_1)$. The result now follows from $s_1 \leq 1$. □

PROPOSITION 11. *Let Γ be a* GF($\sqrt{q}$)*–pencil of Hermitian surfaces with rank sequence $(0, r_1', r_2'+1)$. Then, the base configuration $\mathcal{E}' = \mathcal{E} \cap \pi$ is uniquely determined, according as $V_2 \in \mathcal{E}'$ or not.*

Proof. By Lemma 7, the rank sequence of $\mathcal{E}'$ is (r_1', r_2') and its cardinality is hence determined. Observe that 5 of the 7 classes of [2] are uniquely determined by their rank sequence, the exceptions being classes (d) and (e), both corresponding to the same rank sequence $(0, 1)$. By Lemma 7, Γ in this case has necessarily rank sequence $(0, 0, 2)$ and $|\mathcal{E}| = q^2+1$. Denote then by $\mathcal{C}_1$ and $\mathcal{C}_2$ the two distinct Hermitian cones in Γ and assume they have vertices respectively V_1 and V_2. There are two possibilities for $\mathcal{E}' = \mathcal{E} \cap \pi$:

(1) $\mathcal{E}'$ belongs to class (d), that is $\mathcal{E}'$ consists of $\sqrt{q}-1$ sublines, all disjoint, and 2 more points;
(2) $\mathcal{E}'$ belongs to class (e), that is $\mathcal{E}'$ consists of $\sqrt{q}$ sublines, all concurrent in P.

The former case occurs when $V_1 \notin \mathcal{E}$. If $\mathcal{E}'$ belongs to class (e), then $V_1 = P \in \mathcal{E}$. □

As seen in Proposition 11, there are two cases corresponding to rank sequence $(0, 0, 2)$. If $\mathcal{E}'$ is of class (d), then the pencil is generated by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & a \\ 0 & 0 & a^{\sqrt{q}} & \gamma \end{pmatrix},$$

with $\alpha \neq 1$ and $(\beta - \alpha)(\gamma - \alpha) \neq a^{\sqrt{q}+1}$; if $\mathcal{E}'$ is of class (e), $V_1 \in \mathcal{E}$ but $V_2 \notin \mathcal{E}$ the following generators may be chosen:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & c & 0 \\ 0 & c^{\sqrt{q}} & \alpha & a^{\sqrt{q}} \\ 0 & 0 & a & \alpha \end{pmatrix},$$

with $\alpha \neq 1$ and $a^{\sqrt{q}+1} + c^{\sqrt{q}+1} = 0$ is a set of generators for Γ; finally, if $V_1, V_2 \in \mathcal{E}$, then the pencil may be generated by

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{\sqrt{q}} \\ 0 & 0 & \alpha & 0 \end{pmatrix},$$

with $\alpha \neq 0$.

LEMMA 12. *Suppose that* Γ *contains at least* 3 *distinct cones* $\mathcal{C}_1 \dots \mathcal{C}_2$ *of vertices* $V_1 \dots V_3$. *Then, either the vertices of all the cones in* Γ *are collinear or at most one of them is in* $\mathcal{E}$.

Proof. Suppose $V_1, V_2 \in \mathcal{E}$; then, $V_1V_2 \subseteq \mathcal{E}$. However, for V_1V_2 to be a subset of $\mathcal{E}$, it is necessary for it to be a generator of $\mathcal{C}_3$ also. It follows $V_3 \in V_1V_2$. □

Observe that when Γ contains at least two Hermitian cones, the number of generators fully contained in $\mathcal{E}$ is at most 1.

PROPOSITION 13. *If* Γ *contains* 4 *Hermitian cones, then none of the vertices of such cones belongs to* $\mathcal{E}$ *and exactly* $\sqrt{q}(q - \sqrt{q} - 2)$ *generators of any cone meet* $\mathcal{E}$ *in* $(\sqrt{q} + 1)$ *points, the remaining* $(\sqrt{q} + 1)^2$ *being tangent lines.*

Proof. By Proposition 2, $|\mathcal{E}| = (q - 1)^2$. Suppose $V_1, V_2, V_3, V_4 \in \mathcal{E}$. Then,

$$(q - 1)^2 = (q + 1) + s_2\sqrt{q} + (q\sqrt{q} - s_2),$$

that is $q(q^2 - \sqrt{q} - 3) = s_2(\sqrt{q} - 1)$, a contradiction since $(\sqrt{q} - 1)$ does not divide $q^2 - \sqrt{q} - 3$. If $V_1 \in \mathcal{E}$ while $V_2, V_3, V_4 \notin \mathcal{E}$. Observe that a generator of $\mathcal{C}_1$ either meets a non–degenerate surface $\mathcal{H}$ is 1 or in $\sqrt{q} + 1$ points. Then,

$$(q-1)^2 = s_2\sqrt{q} + 1,$$

which gives $s_2 = q\sqrt{q} - 2\sqrt{q}$, that is $2\sqrt{q} + 1$ generators of $\mathcal{C}$ meet $\mathcal{H}$ in V only and all these generators lie in the tangent plane to $\mathcal{H}$ at V. However, the number of generators of $\mathcal{C}$ on any plane is at most $\sqrt{q} + 1$, which provides a contradiction. It follows that $\mathcal{E}$ does not contain the vertex of any cone in Γ and

$$(q-1)^2 = s_2(\sqrt{q} + 1) + (q\sqrt{q} + 1 - s_2),$$

that is, $s_2 = \sqrt{q}(q - \sqrt{q} - 2)$, which proves the result. □

PROPOSITION 14. *Suppose* Γ *to contain exactly* 3 *cones* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. *Then, there are two possibilities:*

(1) $V_1, V_2, V_3 \notin \mathcal{E}$*: then* $\sqrt{q}(q - \sqrt{q} - 1)$ *components of each cone are* $(\sqrt{q}+1)$*–secants to any non–degenerate Hermitian surface in* Γ*;*
(2) $V_1 \in \mathcal{E}$ *but* $V_2, V_3 \notin \mathcal{E}$*: then* $\sqrt{q}(q-1)$ *generators of the cone* $\mathcal{C}_1$ *meet* $\mathcal{E}$ *in* $(\sqrt{q} + 1)$ *points, the remaining intersecting* $\mathcal{E}$ *in* V_1 *only; the number of generators of the cones* $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *meeting* $\mathcal{E}$ *in* $(\sqrt{q} + 1)$ *points is* $q\sqrt{q} - q - \sqrt{q}$*, the others meeting* $\mathcal{E}$ *in* 1 *point.*

Proof. Suppose $V_i \notin \mathcal{E}$ and let π_i be the polar plane of V_i with respect to a non–singular Hermitian surface $\mathcal{H} \in \Gamma$. Then, $\mathcal{E}'_i = \mathcal{E} \cap \pi_i$ is a configuration of class (b). The cardinality of $\mathcal{E}$ is $q^2 - q + 1$.

(1) Since $V_1 \notin \mathcal{E}$,

$$q^2 - q + 1 = s_2(\sqrt{q} + 1) + (q\sqrt{q} - s_2 + 1).$$

Hence, there are $\sqrt{q}(q - \sqrt{q} - 1)$ components of $\mathcal{C}_1$ meeting $\mathcal{E}$ in $(\sqrt{q} + 1)$ points, the remaining $q + \sqrt{q} + 1$ being tangent to any surface.

(2) Since $V_1 \in \mathcal{E}$, each generator of $\mathcal{C}_1$ meets $\mathcal{E}$ in either 1 or $\sqrt{q} + 1$ points. We get

$$q^2 - q + 1 = 1 + t\sqrt{q}.$$

It follows that $\sqrt{q}(q-1)$ generators through V meet $\mathcal{E}$ in $\sqrt{q} + 1$ points. Consider now another cone $\mathcal{C}_2 \in \Gamma$. Since $V_2 \notin \mathcal{E}$, we get

$$q^2 - q + 1 = s_2(\sqrt{q} + 1) + (q\sqrt{q} + 1 - s_2);$$

hence $s_2 = q\sqrt{q} - q - \sqrt{q}$.

Suppose now $V_1, V_2 \in \mathcal{E}$. Then, the line V_1V_2 is a generator of any surface $\mathcal{H} \in \Gamma$ and we have

$$q^2 - q + 1 = q + 1 + s_2\sqrt{q}.$$

It follows that $s_2 = q\sqrt{q} - 2\sqrt{q}$. This gives that there should be $2\sqrt{q} + 1 > \sqrt{q} + 1$ generators through V_1 meeting $\mathcal{H}$ in V_1 only — a contradiction. □

In the case (1) of Proposition 14 a pencil suitable Γ of rank sequence $(0, 0, 3)$ with $V_1 \notin \mathcal{E}$ is given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & a \\ 0 & 0 & a^{\sqrt{q}} & \gamma \end{pmatrix},$$

with $\alpha \neq 1$ and $(\beta + \gamma)^2 + 4a^{\sqrt{q}+1} = 0$.
For the case (2) of the same proposition the generators have to be of the form.

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & b & 0 & 0 \\ b^{\sqrt{q}} & \alpha & 0 & 0 \\ 0 & 0 & \beta & a \\ 0 & 0 & a^{\sqrt{q}} & \gamma \end{pmatrix},$$

where $(\beta - \gamma)^2 + 4a^{\sqrt{q}+1} = 0$, and $(1 - \alpha)^2 + 4b^{\sqrt{q}+1} \neq 0$ is a square in $\mathrm{GF}(\sqrt{q})$. In the case of rank sequence $(0, 0, 1)$, the configuration $\mathcal{E}'$ is a $q - \sqrt{q} + 1$ arc and Γ is spawned by H_1 and a matrix H_2 such that the minimum polynomial of H_2 is of the form $m(x) = (x - 1) f(x)$ where $f(x)$ is an irreducible polynomial of degree 3.

PROPOSITION 15. *Assume that the only degenerate surface in the pencil Γ is a cone $\mathcal{C}$. Then, either*

(1) *$V \notin \mathcal{E}$ and $\sqrt{q}(q - \sqrt{q} + 1)$ components of $\mathcal{C}$ are meet $\mathcal{E}$ in $\sqrt{q} + 1$ points, or*
(2) *$V \in \mathcal{E}$ and $\mathcal{E}$ contains at least a line.*

Proof. (1) If $V \notin \mathcal{E}$, then $s_1 = 0$ and

$$q^2 + q + 1 = s_2(\sqrt{q} + 1) + (q\sqrt{q} + 1 - s_2);$$

hence, $s_2 = \sqrt{q}(q - \sqrt{q} + 1)$.
(2) If $V \in \mathcal{E}$,

$$q^2 + q + 1 = s_1 q + s_2\sqrt{q} + 1;$$

hence, $s_2 = \sqrt{q}(q + 1 - s_1)$. Since the total number of components of $\mathcal{C}$ is $q\sqrt{q} + 1$, it follows that $s_2 \leq q\sqrt{q} + 1$ and $s_1 \geq 1$. □

Suppose Γ to contain exactly one Hermitian cone whose vertex belongs P to $\mathcal{E}$, and let π be the tangent plane at P to a non–degenerate Hermitian surface in Γ. Clearly, all generators contained in $\mathcal{E}$ have to lie in π; furthermore $s_1 \in \{1, 2, \sqrt{q}+1\}$. Consequently, $s_2 \in \{q\sqrt{q}, q\sqrt{q}-\sqrt{q}, q\sqrt{q}-q\}$. We observe also that if $s_1 > 1$, then all non–degenerate Hermitian surfaces in Γ share the same tangent plane at P.

Finally a pencil associated with the case of Proposition 13 with rank sequence (0, 0, 4) may be generated by the

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

with $\alpha, \beta, \gamma \neq 0, 1$ all distinct elements of $\mathrm{GF}(\sqrt{q})$.

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Luca Giuzzi
Dipartimento di Matematica
Politecnico di Bari
via G. Ameudola 126/B
70126 Bari
Italy
giuzzi@ing.unibs.it

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