Line Polar Grassmann Codes of Orthogonal Type

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Abstract

Polar Grassmann codes of orthogonal type have been introduced in [1]. They are subcodes of the Grassmann code arising from the projective system defined by the Plücker embedding of a polar Grassmannian of orthogonal type. In the present paper we fully determine the minimum distance of line polar Grassmann Codes of orthogonal type for \( q \) odd.

Keywords: Grassmann codes, error correcting codes, line Polar Grassmannians.

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1. Introduction

Codes \( C_{m,k} \) arising from the Plücker embedding of the \( k \)-Grassmannians of \( m \)-dimensional vector spaces have been widely investigated since their first introduction in [10, 11]. They are a remarkable generalization of Reed–Muller codes of the first order and their monomial automorphism groups and minimum weights are well understood, see [8, 5, 6, 4].

In [1], the first two authors of the present paper introduced some new codes \( P_{n,k} \) arising from embeddings of orthogonal Grassmannians \( \Delta_{n,k} \). These codes correspond to the projective system determined by the Plücker embedding of the Grassmannian \( \Delta_{n,k} \) representing all totally singular \( k \)-spaces with respect to some non-degenerate quadratic form \( \eta \) defined on a vector space \( V(2n + 1, q) \) of dimension \( 2n + 1 \) over a finite field \( \mathbb{F}_q \). An orthogonal Grassmann code \( P_{n,k} \) can be obtained from the ordinary Grassmann code \( G_{2n+1,k} \) by just deleting all the columns corresponding to \( k \)-spaces which are non-singular with respect to \( \eta \); it is thus a punctured version of \( G_{2n+1,k} \). For \( q \) odd, the dimension of \( P_{n,k} \) is the same as that of \( G_{2n+1,k} \), see [1]. The minimum distance \( d_{\text{min}} \) of \( P_{n,k} \) is always bounded away from 1. Actually, it has been shown in [1] that for \( q \) odd, \( d_{\text{min}} \geq q^{k(n-k)+1} + q^{k(n-k)} - q \). By itself, this proves that the redundancy of these codes is somehow better than that of \( G_{2n+1,k} \).

In the present paper we prove the following theorem, fully determining all the parameters for the case of line orthogonal Grassmann codes (that is orthogonal polar Grassmann codes with \( k = 2 \)) for \( q \) odd.

Main Theorem. For \( q \) odd, the minimum distance \( d_{\text{min}} \) of the orthogonal Grassmann code \( P_{n,2} \) is

\[ d_{\text{min}} = q^{4n-5} - q^{3n-4}. \]

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Furthermore, for \( n > 2 \) all words of minimum weight are projectively equivalent; for \( n = 2 \) there are two different classes of projectively equivalent minimum weight codewords.

Hence, we have the following.

**Corollary 1.1.** For \( q \) odd, line polar Grassmann codes of orthogonal type are \([N, K, d_{\text{min}}]\)-projective codes with

\[
N = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q^2 - 1)(q - 1)}, \quad K = \binom{2n + 1}{2}, \quad d_{\text{min}} = q^{4n-5} - q^{3n-4}.
\]

### 1.1. Organization of the paper

In Section 2 we recall some well–known facts on projective systems and related codes, as well as the notion of polar Grassmannian of orthogonal type. In Section 3 we prove our main theorem.

## 2. Preliminaries

### 2.1. Projective systems and Grassmann codes

An \([N, K, d_{\text{min}}]_q\) projective system \( \Omega \subseteq \text{PG}(K - 1, q) \) is a set of \( N \) points in \( \text{PG}(K - 1, q) \) such that there is a hyperplane \( \Sigma \) of \( \text{PG}(K - 1, q) \) with \#(\( \Omega \setminus \Sigma \)) = \( d_{\text{min}} \) and for any hyperplane \( \Sigma' \) of \( \text{PG}(K - 1, q) \),

\[#(\Omega \setminus \Sigma') \geq d_{\text{min}}.\]

Existence of \([N, K, d_{\text{min}}]_q\) projective systems is equivalent to that of projective linear codes with the same parameters; see, for instance, [12]. Indeed, let \( \Omega \) be a projective system and denote by \( G \) a matrix whose columns \( G_1, \ldots, G_N \) are the coordinates of representatives of the points of \( \Omega \) with respect to some fixed reference system. Then, \( G \) is the generator matrix of an \([N, K, d_{\text{min}}]\) code over \( \mathbb{F}_q \), say \( C = C(\Omega) \). The code \( C(\Omega) \) is not, in general, uniquely determined, but it is unique up to code equivalence. We shall thus speak, with a slight abuse of language, of the code defined by \( \Omega \).

As any word \( c \) of \( C(\Omega) \) is of the form \( c = mG \) for some row vector \( m \in \mathbb{F}_q^K \), it is straightforward to see that the number of zeroes in \( c \) is the same as the number of points of \( \Omega \) lying on the hyperplane \( \Pi_c \) of equation \( m \cdot x = 0 \), where \( m \cdot x = \sum_{i=1}^{K} m_i x_i \) and \( m = (m_i)_1^K \), \( x = (x_i)_1^K \). The weight (i.e. the number of non–zero components) of \( c \) is then

\[
\text{wt}(c) := |\Omega| - |\Omega \cap \Pi_c|.
\]

Thus, the minimum distance \( d_{\text{min}} \) of \( C \) is

\[
d_{\text{min}} = |\Omega| - f_{\text{max}}, \quad \text{where} \quad f_{\text{max}} = \max_{\Sigma \leq \text{PG}(K-1, q), \dim \Sigma = K-2} |\Omega \cap \Sigma|.
\]

We point out that any projective code \( C(\Omega) \) can also be regarded, equivalently, as an evaluation code over \( \Omega \) of degree 1. In particular, when \( \Omega \) spans the whole of \( \text{PG}(K - 1, q) = \text{PG}(W) \), with \( W \) the underlying vector space, then there is a bijection, induced by the standard inner product of \( W \), between the points of the dual vector space \( W^* \) and the codewords \( c \) of \( C(\Omega) \).

Let \( G_{2n+1,k} \) be the Grassmannian of the \( k \)-subspaces of a vector space \( V := V(2n + 1, q) \), with \( k \leq n \) and let \( \eta : V \to \mathbb{F}_q \) be a non-degenerate quadratic form over \( V \).

Denote by \( \varepsilon_k : G_{2n+1,k} \to \text{PG}(\bigwedge^k V) \) the usual Plücker embedding

\[
\varepsilon_k : \text{Span}(v_1, \ldots, v_k) \to \text{Span}(v_1 \wedge \cdots \wedge v_k).
\]
The orthogonal Grassmannian $\Delta_{n,k}$ is a geometry having as points the $k$–subspaces of $V$ totally singular for $\eta$. Let $\varepsilon_k(G_{2n+1,k}) := \{\varepsilon_k(X_k) : X_k$ is a point of $G_{2n+1,k}\}$ and $\varepsilon_k(\Delta_{n,k}) = \{\varepsilon_k(\tilde{X}_k) : \tilde{X}_k$ is a point of $\Delta_{n,k}\}$. Clearly, we have $\varepsilon_k(\Delta_{n,k}) \subseteq \varepsilon_k(G_{2n+1,k}) \subseteq \text{PG}(\wedge^k V)$. Throughout this paper we shall denote by $\mathcal{F}_{n,k}$ the code arising from the projective system $\varepsilon_k(\Delta_{n,k})$. By [3, Theorem 1.1], if $n \geq 2$ and $k \in \{1, \ldots, n\}$, then $\dim \text{Span}(\varepsilon_k(\Delta_{n,k})) = \left(\begin{array}{c} 2n+1 \\ k \end{array}\right)$ for odd, while $\dim \text{Span}(\varepsilon_k(\Delta_{n,k})) = \left(\begin{array}{c} 2n+1 \\ k \end{array}\right) - \left(\begin{array}{c} 2n+1 \\ k-2 \end{array}\right)$ when $q$ is even.

We recall that for $k < n$, any line of $\Delta_{n,k}$ is also a line of $G_{2n+1,k}$. For $k = n$, the lines of $\Delta_{n,n}$ are not lines of $G_{2n+1,n}$; indeed, in this case $\varepsilon_n|_{\Delta_{n,n}} : \Delta_{n,n} \rightarrow \text{PG}(\wedge^n V)$ maps the lines of $\Delta_{n,n}$ onto non-singular conics of $\text{PG}(\wedge^n V)$.

The projective system identified by $\varepsilon_k(\Delta_{n,k})$ determines a code of length $N = \prod_{i=0}^{k-1} \frac{2^{q(n-i)} - 1}{q}$ and dimension $K = \left(\begin{array}{c} 2n+1 \\ k \end{array}\right)$ or $K = \left(\begin{array}{c} 2n+1 \\ k \end{array}\right) - \left(\begin{array}{c} 2n+1 \\ k-2 \end{array}\right)$ according to whether $q$ is odd or even. The following universal property provides a well–known characterization of alternating multilinear forms; see for instance [9, Theorem 14.23].

**Theorem 2.1.** Let $V$ and $U$ be vector spaces over the same field. A map $f : V^k \rightarrow U$ is alternating $k$–linear if and only if there is a linear map $\hat{f} : \wedge^k V \rightarrow U$ with $\hat{f}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = f(v_1, v_2, \ldots, v_k)$. The map $\hat{f}$ is uniquely determined.

In general, the dual space $(\wedge^k V)^* \cong \wedge^k V^*$ of $\wedge^k V$ is isomorphic to the space of all $k$–linear alternating forms of $V$. For any given non-null vector $v \in \wedge^{2n+1-k} V \cong V(1,q) \cong \mathbb{F}_q$, we have an isomorphism $\gamma_v : \wedge^{2n+1-k} V \rightarrow (\wedge^k V)^*$ defined by $\gamma_v(\omega)(x) = c$ for any $\omega \in \wedge^{2n+1-k}$ and $x \in \wedge^k V$, where $c \in \mathbb{F}_q$ is such that $\omega \wedge x = cv$. Clearly, as $v \neq 0$ varies in $\wedge^{2n+1-k} V$ we obtain different isomorphisms. For the sake of simplicity, we will say that $\omega \in \wedge^{2n+1-k} V$ acts on $x \in \wedge^k V$ as $\omega \wedge x$.

For any $k = 1, \ldots, 2n$ and $\varphi \in (\wedge^k V)^*$, $v \in \wedge^k V$ we shall use the symbol $\langle \varphi, v \rangle$ to denote the bilinear pairing

$$\langle \varphi, v \rangle = \varphi(v).$$

Since the codewords of $\mathcal{F}_{n,k}$ bijectively correspond to functionals on $\wedge^k V$, we can regard a codeword as an element of $(\wedge^k V)^* \cong \wedge^k V^*$.

In this paper we are concerned with line Grassmannians, that is we assume $k = 2$.

By Theorem 2.1, we shall implicitly identify any functional $\varphi \in (\wedge^2 V)^*$ with the (necessarily degenerate) alternating bilinear form

$$\begin{align*}
\{ V \times V & \rightarrow \mathbb{F}_q \\
(x, y) & \rightarrow \varphi(x \wedge y).
\end{align*}$$

The **radical** of $\varphi$ is the set

$$\text{Rad}(\varphi) := \{ v \in V : \forall w \in V, \varphi(v, w) = 0 \}.$$

This is always a vector space and its codimension in $V$ is even. As $\dim V$ is odd, $2n - 1 \geq \dim \text{Rad}(\varphi) \geq 1$ for $\varphi \neq 0$.

We point out that it has been proved in [8] that the minimum weight codewords of the line projective Grassmann code $G_{2n+1,2}$, correspond to points of $\varepsilon_{2n-1}(G_{2n+1,2n-1})$; these can be regarded as non-null bilinear alternating forms of $V$ of maximum radical. Actually, non-null bilinear forms of maximum radical may yield minimum weight codewords also for Symplectic Polar Grassmann Codes, see [2].
In the case of orthogonal line Grassmannians, not all points of $G_{2n+1,2n-1}$ yield codewords of $P_{n,2}$ of minimum weight. However, as a consequence of the proof of our main result, we shall see that for $n > 2$ all the codewords of minimum weight of $P_{n,2}$ do indeed correspond to some $(2n-1)$-dimensional subspaces of $V$, that is to say, to bilinear alternating forms of maximum radical. In the case $n = 2$, there are two classes of minimum weight codewords: one corresponding to bilinear alternating forms of maximum radical and another corresponding to certain bilinear alternating forms with radical of dimension 1.

2.2. A recursive condition

Since $\Lambda^k V^* \cong (\Lambda^k V)^* \cong \Lambda^{2n+1-k} V$, for any $\varphi \in (\Lambda^k V)^*$ there is an element $\hat{\varphi} \in \Lambda^{2n+1-k} V$ such that
\[
\langle \varphi, x \rangle = \hat{\varphi} \wedge x, \quad \forall x \in \Lambda^k V.
\]

Fix now $u \in V$ and $\varphi \in (\Lambda^k V)^*$. Then, there is a unique element $\varphi_u \in \Lambda^{k-1} V^*$ such that $\hat{\varphi}_u = \hat{\varphi} \wedge u \in \Lambda^{2n+2-k} V$. 

Let $Q$ be the parabolic quadric defined by the (non-degenerate) quadratic form $\eta$. For any $u \in Q$, put $V_u := u^\perp Q/\Span(u)$. Observe that as $\langle \varphi_u, u \wedge w \rangle = \hat{\varphi} \wedge u \wedge u \wedge w = 0$ for any $u \wedge w \in \Lambda^{k-1} V$, the functional $\varphi_u : \Lambda^{k-1} V_u \rightarrow F_q$
\[
\left\{ \begin{array}{ll}
\Lambda^{k-1} V_u & \rightarrow F_q \\
x + (u \Lambda^{k-2} V) & \rightarrow \varphi_u(x)
\end{array} \right.
\]

with $x \in \Lambda^{k-1} V$ and $u \Lambda^{k-2} V := \{ u \wedge y : y \in \Lambda^{k-2} V \}$ is well defined. Furthermore, $V_u$ is endowed with the quadratic form $\eta_u : x + \Span(u) \rightarrow \eta(x)$. Clearly, $\dim V_u = 2n - 1$. It is well known that the set of all totally singular points for $\eta_u$ is a parabolic quadric of rank $n-1$ in $V_u$ which we shall denote by $\Res Q u$. In other words the points of $\Res Q u$ are the lines of $Q$ through $u$.

We are now ready to deduce a recursive relation on the weight of codewords, in the spirit of [8].

**Lemma 2.2.** Let $\varphi \in \Lambda^k V^*$. Then,
\[
\wt(\varphi) = \frac{1}{q^k - 1} \sum_{u \in Q} \wt(\varphi_u).
\]

**Proof.** Recall that
\[
\wt(\varphi) = \# \{ \Span(v_1, \ldots, v_k) : \langle \varphi, v_1 \wedge \cdots \wedge v_k \rangle \neq 0, \Span(v_1, \ldots, v_k) \in \Delta_{n,k} \} = 
\frac{1}{|\text{GL}_k(q)|} \# \{ (v_1, \ldots, v_k) : \langle \varphi, v_1 \wedge \cdots \wedge v_k \rangle \neq 0, \Span(v_1, \ldots, v_k) \in \Delta_{n,k} \}
\]

where the list $(v_1, \ldots, v_k)$ is an ordered basis of $\Span(v_1, \ldots, v_k) \subset Q$.

For any point $u \in Q$, we have $\Span(u, v_2, \ldots, v_k) \in \Delta_{n,k}$ if and only if $\Span_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\Res Q u)$, where $\Delta_{n-1,k-1}(\Res Q u)$ is the $(k-1)$-Grassmannian of $\Res Q u$ and by the symbol $\Span_u(v_2, \ldots, v_k)$ we mean $\Span(u, v_2, \ldots, v_k)/\Span(u)$. Furthermore, given a space $\Span_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\Res Q u)$, any of the $q^{k-1}$ lists $(u, v_2 + \alpha_2 u, \ldots, v_k + \alpha_k u)$ is a basis for the same totally singular $k$–space through $u$, namely $\Span(u, v_2, \ldots, v_k)$. Conversely, given
any totally singular $k$–space $W \in \Delta_{n,k}$ with $u \in W$ there are $v_2, \ldots, v_k \in \text{Res}_u$ such that $W = \text{Span}(u, v_2, \ldots, v_k)$ and $\text{Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)$. Let

$$\Omega_u := \{(u, v_2 + \alpha u, \ldots, v_k + \alpha u) : \langle \varphi, u \wedge v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u), \alpha_2, \ldots, \alpha_k \in \mathbb{F}\}.$$  

Then, we have the following disjoint union

$$\{(v_1, \ldots, v_k) : \langle \varphi, v_1 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}(v_1, \ldots, v_k) \in \Delta_{n,k}\} = \bigcup_{u \in \mathbb{Q}} \Omega_u. \quad (4)$$

Observe that if $u$ is not singular, then, $\Omega_u = \emptyset$, as $\text{Span}(u, v_2, \ldots, v_k) \not\subseteq \mathbb{Q}$; likewise, if $\varphi_u = 0$, then, $\langle \varphi_u, v_2 \wedge \cdots \wedge v_k \rangle = 0$ for any $v_2, \ldots, v_k$ and, consequently, $\Omega_u = \emptyset$.

The coefficients $\alpha_i$, $2 \leq i \leq k$, are arbitrary in $\mathbb{F}$; thus,

$$\#\Omega_u = q^{k-1} \# \{(u, v_2, \ldots, v_k) : \langle \varphi, v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)\}.$$  

Hence,

$$|\text{GL}_k(q)|\text{wt}(\varphi) = \sum_{u \in \mathbb{Q}} \#\Omega_u = q^{k-1} \sum_{u \in \mathbb{Q}, \varphi_u \neq 0} \# \{(u, v_2, \ldots, v_k) : \langle \varphi, v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)\}. \quad (5)$$

Since $u$ is fixed,

$$\# \{(u, v_2, \ldots, v_k) : \langle \varphi, v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)\} = \# \{(v_2, \ldots, v_k) : \langle \varphi, v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)\}.$$  

On the other hand, by (3) and by the definition of $\varphi_u$,

$$|\text{GL}_{k-1}(q)|\text{wt}(\varphi_u) = \# \{(v_2, \ldots, v_k) : \langle \varphi_u, v_2 \wedge \cdots \wedge v_k \rangle \neq 0, \text{ Span}_u(v_2, \ldots, v_k) \in \Delta_{n-1,k-1}(\text{Res}_u)\};$$

thus,

$$\text{wt}(\varphi) = q^{k-1} |\text{GL}_{k-1}(q)| \sum_{u \in \mathbb{Q}, \varphi_u \neq 0} \text{wt}(\varphi_u) = \frac{1}{q^k - 1} \sum_{u \in \mathbb{Q}, \varphi_u \neq 0} \text{wt}(\varphi_u). \quad (6)$$

\hfill \Box

3. Proof of the Main Theorem

As dim $V$ is odd, all non–degenerate quadratic forms on $V$ are projectively equivalent. For the purposes of the present paper we can assume without loss of generality that a basis $(e_1, \ldots, e_{2n+1})$ has been fixed such that

$$\eta(x) := \sum_{i=1}^n x_{2i-1}x_{2i} + x_{2n+1}^2. \quad (7)$$
Let \( \beta(x, y) := \eta(x + y) - \eta(x) - \eta(y) \) be the bilinear form associated with \( \eta \). As in Section 2.2, denote by \( \mathcal{Q} \) the set of the non-zero totally singular vectors for \( \eta \). Clearly, for any \( k \)-dimensional vector subspace \( W \) of \( V \), then \( W \in \Delta_{n,k} \) if and only if \( W \subseteq \mathcal{Q} \).

Henceforth we shall work under the assumption \( k = 2 \). Denote by \( \varphi \) an arbitrary alternating bilinear form defined on \( V \) and let \( M \) and \( S \) be the matrices representing respectively \( \beta \) and \( \varphi \) with respect to the basis \( (e_1, \ldots, e_{2n+1}) \) of \( V \). Write \( \perp_{\mathcal{Q}} \) for the orthogonal relation induced by \( \eta \) and \( \perp_W \) for the (degenerate) symplectic relation induced by \( \varphi \). In particular, for \( v \in V \), the symbols \( v^\perp_{\mathcal{Q}} \) and \( v^\perp_W \) will respectively denote the space orthogonal to \( v \) with respect to \( \eta \) and \( \varphi \). Likewise, when \( X \) is a subspace of \( V \), the notations \( X^\perp_{\mathcal{Q}} \) and \( X^\perp_W \) will be used to denote the spaces orthogonal to \( X \) with respect to \( \eta \) and \( \varphi \). We shall say that a subspace \( X \) is totally singular if \( X \subseteq X^\perp_{\mathcal{Q}} \) and totally isotropic if \( X \subseteq X^\perp_W \).

**Lemma 3.1.** Let \( \mathcal{Q} \) be a parabolic quadric with equation of the form (7), and let \( p \in V \), \( p \neq 0 \). Denote by \( \rho \) a codeword corresponding to the hyperplane \( p^\perp_{\mathcal{Q}} \). Then,

\[
\text{wt}(\rho) = \begin{cases} 
q^{2n-1} & \text{if } \eta(p) = 0 \\
q^{2n-1} - q^{n-1} & \text{if } \eta(p) \text{ is a non-zero square} \\
q^{2n-1} + q^{n-1} & \text{if } \eta(p) \text{ is a non-square.}
\end{cases}
\]

**Proof.** If \( \eta(p) = 0 \), then \( p \in \mathcal{Q} \) and \( p^\perp_{\mathcal{Q}} \cap \mathcal{Q} \) is a cone with basis a parabolic quadric of rank \( n - 1 \); it has \( 1 + (q^{2n-1} - q)/(q - 1) \) projective points, see [7]. The value of \( \text{wt}(\rho) \) now directly follows from (1).

Suppose now \( p \) to be external to \( \mathcal{Q} \), that is \( p^\perp_{\mathcal{Q}} \cap \mathcal{Q} \) is a hyperbolic quadric; it is immediate to see that in this case \( \text{wt}(\rho) = q^{2n-1} - q^{n-1} \). Likewise, when \( p \) is internal to \( \mathcal{Q} \), \( \text{wt}(\rho) = q^{2n-1} + q^{n-1} \).

The orthogonal group \( O(V) \) stabilizing the quadric \( \mathcal{Q} \) has 3 orbits on the points of \( V \); these correspond respectively to totally singular, external and internal points to \( \mathcal{Q} \). By construction, all elements in the same orbit are isometric 1-dimensional quadratic spaces. In other words, the quadratic class of \( \eta(p) \) is constant on each of these orbits. In particular, the point \( e_{2n+1} \) is totally isotropic if \( \eta(e_{2n+1}) = 1 \) is a square. Thus we have that external points to \( \mathcal{Q} \) correspond to those \( p \) for which \( \eta(p) \) is a square, \( \eta(p) \neq 0 \) and internal points correspond to those for which \( \eta(p) \) is a non-square. \( \square \)

### 3.1. Some linear algebra

**Lemma 3.2.**

1. For any \( v \in V \), \( v^\perp_{\mathcal{Q}} = v^\perp_W \) if and only if \( v \) is an eigenvector of non-zero eigenvalue of \( T := M^{-1}S \).

2. The radical \( \text{Rad}(\varphi) \) of \( \varphi \) corresponds to the eigenspace of \( T \) of eigenvalue \( 0 \).

**Proof.**

1. Observe that \( v^\perp_{\mathcal{Q}} = v^\perp_W \) if and only if the equations \( x^TMv = 0 \) and \( x^TSv = 0 \) are equivalent for any \( x \in V \). This means that there exists an element \( \lambda \in F_q \setminus \{0\} \) such that \( Sv = \lambda Mv \). As \( M \) is non-singular, the latter says that \( v \) is an eigenvector of non-zero eigenvalue \( \lambda \) for \( T \).

2. Let \( v \) be an eigenvector of \( T \) of eigenvalue \( 0 \). Then \( M^{-1}Sv = 0 \), hence \( Sv = 0 \) and \( x^TSv = 0 \) for every \( x \in V \), that is \( v^\perp_W = V \). This means \( v \in \text{Rad}(\varphi) \). \( \square \)

We can now characterize the eigenspaces of \( T \).

**Lemma 3.3.** Let \( \mu \) be a non-zero eigenvalue of \( T \) and \( V_\mu \) be the corresponding eigenspace. Then,

1. \( \forall v \in V_\mu \) and \( r \in \text{Rad}(\varphi) \), \( r \perp_{\mathcal{Q}} v \). Hence, \( V_\mu \leq r^\perp_{\mathcal{Q}} \).
(2) The eigenspace $V_\mu$ is both totally isotropic for $\varphi$ and totally singular for $\eta$.

(3) Let $\lambda, \mu \neq 0$ be two not necessarily distinct eigenvalues of $T$ and $u, v$ be two corresponding eigenvectors. Then, one of the following holds:
   
   (a) $u \perp Q v$ and $u \perp W v$.
   
   (b) $\mu = -\lambda$.

(4) If $\lambda$ is an eigenvalue of $T$ then $-\lambda$ is an eigenvalue of $T$.

Proof. 1. Take $v \in V_\mu$. As $Tv = M^{-1}Sv = \mu v$ we also have $\mu v = v^T \lambda T M^{-1}$. So, $v^T M = -\mu v^T \lambda T$. Let $r \in \text{Rad}(\varphi)$. Then, as $S^T = -S$, $v^T Mr = \mu v^T \lambda T r$ and $v^T Sr = 0$ for any $v$, we have $v^T Mr = 0$, that is $r \perp Q v$.

2. Let $v \in V_\mu$. Then $M^{-1}Sv = \mu v$, which implies $Sv = \mu Mv$. Hence, $v^T Sv = \mu v^T Mv$. Since $v^T Sv = 0$ and $\mu \neq 0$, we also have $v^T Mv = 0$, for every $v \in V_\mu$. Thus, $V_\mu$ is totally singular for $\eta$. Since $V_\mu$ is totally singular, for any $u \in V_\mu$ we have $u^T Mv = 0$; so, $u^T S v = \mu u^T M v = 0$, that is $V_\mu$ is also totally isotropic.

3. Suppose that either $u \perp Q v$ or $u \perp W v$. Since, by Lemma 3.2, $u \perp_Q = u \perp_W$ and $v \perp_Q = v \perp_W$, we have $Mu = \lambda^{-1} Su$ and $M v = \mu^{-1} Sv$. So, $u \perp_Q v$ or $u \perp_W v$ implies $v^T Mu \neq 0 \neq v^T Sw$. Since $M^{-1} Su = \lambda u$ and $M^{-1} Sw = \mu v$, we have

$$v^T Su = v^T S (\lambda^{-1} M^{-1} Su) = \lambda^{-1} (M^{-1} Sw) v = -(\lambda^{-1} \mu) v^T Sw;$$

hence, $-\lambda^{-1} \mu = 1$.

4. Let $\lambda \neq 0$ be an eigenvalue of $T$ and $x$ a corresponding eigenvector. Then $M^{-1} S x = \lambda x$ if and only if $S M^{-1} S x = \lambda S x$, which, in turn, is equivalent to $-(M^{-1} S)^T S x = \lambda S x$, that is $(M^{-1} S)^T (S x) = -\lambda S x$. Since $\lambda \neq 0$, $S x$ is an eigenvector of $(M^{-1} S)^T$ of eigenvalue $-\lambda$. Clearly, $(M^{-1} S)^T$ and $M^{-1} S$ have the same eigenvalues, so $-\lambda$ is an eigenvalue of $T$.

Corollary 3.4. Let $V_\lambda$ and $V_\mu$ be two eigenspaces of non-zero eigenvalues $\lambda \neq -\mu$. Then, $V_\lambda \oplus V_\mu$ is both totally singular and totally isotropic.

3.2. Minimum weight codewords

Recall that $\varphi \in \bigwedge^2 V^*$ and, for any $u \in Q$, $\varphi_u \in V^*$. In particular, $\varphi_u$ either determines a hyperplane of $V_\mu = u^T \varphi / \text{Span}(u)$ or it is null on $V_\mu$.

Lemma 3.5. $\varphi_u = 0$ if and only if $u$ is an eigenvector of $T$.

Proof. By Lemma 3.2, $u$ is an eigenvector of $T$ if and only if $u \perp Q \subseteq u \perp W$. By definition of $\perp Q$, for every $v \in u \perp Q \cap Q$, we have $\text{Span}(u, v) \in \Delta_{n-2}$. However, as $v \in u \perp W$, also $\langle \varphi, u \wedge v \rangle = 0$. So, $\varphi_u(v) = 0$, $\forall v \in u \perp Q$. Thus, $\varphi_u = 0$ on $\text{Res}_{Q} u$. Conversely, reading the argument backwards, we see that if $\varphi_u = 0$ then $u$ is eigenvector of $T$.

We remark that $\varphi_u = 0$ if and only if $u \in \ker T$ (by Lemma 3.2(2)).

Lemma 3.6. Suppose $u \in Q$ not to be an eigenvector of $T$. Then,

$$\text{wt}(\varphi_u) = \begin{cases} q^{2n-3} & \text{if } \eta(Tu) = 0 \\ q^{2n-3} - q^{n-2} & \text{if } \eta(Tu) \neq 0 \text{ is a square} \\ q^{2n-3} + q^{n-2} & \text{if } \eta(Tu) \text{ is a non-square} \end{cases}$$
Proof. Let \( a_u := Tu \) and let \( Q_u := a_u^{-1} Q \cap Q \). Note that \( u \in Q_u \cap u^{-1} Q \). Indeed, \( u^T M u = u^T S u = 0 \). So, \( \text{wt} (\bar{\sigma}_u) = \text{wt} (\bar{\sigma}_{a_u}) \). The quadric \( \text{Res}_{Q,u} := (Q_u \cap u^{-1} Q)/\text{Span}(u) \) is either hyperbolic, elliptic or degenerate according as \( a_u \) is external, internal or contained in \( Q \). The result now follows from Lemma 3.1.

Define
\[
\mathcal{A}' := \{ u : u \in Q \text{ and } u \text{ non-eigenvector of } T \}, \quad A' := \# \mathcal{A}';
\]
\[
\mathcal{B} := \{ u : u \in \mathcal{A}' \text{ and } Tu \in Q \}, \quad B := \# \mathcal{B};
\]
\[
\mathcal{C} := \{ u : u \in \mathcal{A}' \text{ and } \eta(Tu) \text{ is a non-square} \}, \quad C := \# \mathcal{C}.
\]
By definition, both \( \mathcal{B} \) and \( \mathcal{C} \) are subset of \( \mathcal{A}' \). Using (6) we can write
\[
\text{wt}(\phi) = \frac{q^{2n-3} - q^{n-2}}{q^2 - 1} A' + \frac{q^{n-2}}{q^2 - 1} B + \frac{2q^{n-2}}{q^2 - 1} C.
\] (8)

Put \( A = q^{2n-2} - 1 - \# \{ u : u \in Q \text{ and } u \text{ eigenvector of } T \} \); then, (8) becomes
\[
\text{wt}(\phi) = q^{4n-5} - q^{3n-4} + \frac{q^{n-2}}{q^2 - 1} ((q^{n-1} - 1)A + B + 2C).
\] (9)

Clearly, \( B, C \geq 0 \). We investigate \( A \) more closely. Let \( \text{Spec}'(T) \) be the set of non-zero eigenvalues of \( T \) and let \( V_\lambda = \ker(T - \lambda I) \) be the corresponding eigenspaces for \( \lambda \in \text{Spec}'(T) \). By Lemma 3.3, each space \( V_\lambda \) is totally singular; thus
\[
A = q^{2n-2} - 1 - \sum_{\lambda \in \text{Spec}'(T)} (\# V_\lambda - 1) = \#(\ker T \cap Q).
\] (10)

Let \( r \in \mathbb{N} \) be such that \( \dim \text{Rad}(\phi) = \dim \ker T = 2(n - r) + 1 \), where by Theorem 2.1, we may regard \( \phi \) as a bilinear alternating form.

The non-degenerate symmetric bilinear form \( \beta \) induces a symmetric bilinear form \( \beta^* \) on \( V^* \), defined as \( \beta^*(v_1^*, v_2^*) = \beta(v_1, v_2) \) where \( v_1^*, v_2^* \) are functionals determining respectively the hyperplanes \( v_1^* \cap Q \) and \( v_2^* \cap Q \). In particular, the given basis \( (e_1, \ldots, e_{2n+1}) \) of \( V \), the above correspondence determines a basis \( (e_1, \ldots, e_{2n+1}) \) of \( V^* \), where \( e_1 \), as a functional, describes the hyperplane \( e_i^* \) for \( 1 \leq i \leq 2n + 1 \). As before, let also \( O(V) \) be the orthogonal group stabilizing \( Q \). We have the following theorem.

**Theorem 3.7.** For any \( \phi \in \bigwedge^2 V^* \) exactly one of the following conditions holds:

1. \( r = 1 \); then \( \text{wt}(\phi) \geq q^{4n-5} - q^{3n-4} \) with equality occurring if and only if \( \phi \) is in the \( O(V) \)-orbit of \( e_1 \land e_{2n+1} \);

2. \( r > 1 \) and \( A > 0 \): in this case \( \text{wt}(\phi) > q^{4n-5} - q^{3n-4} \);

3. \( r > 1 \) and \( A < 0 \): in this case \( r = n = 2 \) and \( \phi \) is in the \( O(V) \)-orbit of \( e_1 \land e_2 + e_3 \land e_4 \) with \( \text{wt}(\phi) = q^3 - q^2 \).

Proof. If \( r = 1 \), then \( \dim \text{Rad}(\phi) = 2n - 1 \). As \( \phi \in \bigwedge^2 V^* \) has tensor rank 1 (i.e. is fully decomposable), \( \phi \) determines a unique 2-dimensional subspace \( W_\phi \) of \( V^* \). In particular, the subspace \( W_\phi \) is endowed with the quadratic form obtained from the restriction of \( \beta^* \) to \( W_\phi \). There are just 5 types of 2-dimensional quadratic spaces; they correspond respectively to the forms \( f_1(x, y) = 0, f_2(x, y) = y^2, f_3(x, y) = \varepsilon y^2, f_4(x, y) = x^2 - \varepsilon y^2 \) and \( f_5(x, y) = xy \), where \( \varepsilon \) is a non-square in \( \mathbb{F}_q \) and the coordinates are with respect to a given reference system of \( W_\phi \).
For each $f_i$, $1 \leq i \leq 5$, there are some $\varphi_i \in \bigwedge^2 V^*$ such that $\beta^*|_{W_{\varphi_i}} \cong f_i$. Examples of such $\varphi_i$ inducing, respectively, $f_i$ for $i = 1, \ldots, 5$ are the following: $\varphi_1 = e^1 \wedge e^3$, $\varphi_2 = e^1 \wedge e^{2n+1}$, $\varphi_3 = e^1 \wedge (e^3 + \varepsilon e^4)$, $\varphi_4 = e^{2n+1} \wedge (e^1 - \varepsilon e^2)$ and $\varphi_5 = e^1 \wedge e^2$.

Using Witt’s extension theorem we see that there always is an isometry between a given $W_{\varphi}$ and any of these spaces $W_{\varphi_i}$ ($1 \leq i \leq 5$) which can be extended to an element of $O(V)$. In other words any form with $r = 1$ is equivalent to one of the aforementioned five elements of $\bigwedge^2 V^*$.

A direct computation shows that the list of possible weights is as follows:

\[
\begin{align*}
\text{wt}(e^1 \wedge e^2) &= \text{wt}(e^{2n+1} \wedge (e^1 - \varepsilon e^2)) = q^{n-5} - q^{2n-3}, \\
\text{wt}(e^1 \wedge e^3) &= q^{n-5}, \\
\text{wt}(e^1 \wedge e^{2n+1}) &= q^{n-3} - q^{4n-4}, \\
\text{wt}(e^1 \wedge (e^3 + \varepsilon e^4)) &= q^{n-5} + q^{3n-4}.
\end{align*}
\]

As an example we will explicitly compute $\text{wt}(e^1 \wedge e^2)$. The remaining cases are analogous. Since $\varphi_5 = e^1 \wedge e^2$, we have, by (3),

\[
\text{wt}(\varphi_5) = \# \{(v_1, v_2): v_1, v_2 \in \{e_1, e_2\}^\perp \cap Q, \beta(e_1 + v_1, e_2 + v_2) = 0 \}.
\]

In particular, as

\[
\beta(e_1 + v_1, e_2 + v_2) = \beta(e_1, v_2) + \beta(v_1, e_2) + \beta(e_1, e_2) + \beta(v_1, v_2) = 1 + \beta(v_1, v_2)
\]

we have $\beta(v_1, v_2) = -1$. Observe that $Q' := \{e_1, e_2\}^\perp \cap Q$ is a non-singular parabolic quadric $Q(2n - 2, q)$ of rank $n - 1$; thus it contains $(q^{2n-2} - 1) n$ non-zero vectors and we can choose $v_1$ in $(q^{2n-2} - 1)$ ways. For each projective point $p \in Q'$ with $p \notin v_1^\perp$ there is exactly one vector $v_2$ such that $v_2 \notin p$ and $\beta(v_1, v_2) = -1$. The number of such points is

\[
\# Q' - \#(v_1^\perp \cap Q') = q^{2n-2} - 1 - (q^{2n-2} - 1) = q^{2n-3}.
\]

In particular, the overall weight of $\text{wt}(\varphi_5)$ is

\[
\text{wt}(\varphi_5) := q^{2n-3}(q^{2n-2} - 1) = q^{4n-5} - q^{2n-3}.
\]

The case $e^1 \wedge e^{2n+1}$ will yield words of minimum weight.

Suppose now $r > 1$. Clearly,

\[
\# \text{ker}(T) \cap Q \leq \# \text{ker}(T) - 1 = q^{2n-2r+1} - 1.
\]

Furthermore, if $\lambda \in \text{Spec}'(T)$ then also $-\lambda \in \text{Spec}'(T)$ by Lemma 3.3 (4). Thus, we can write $\text{Spec}'(T) = \{\lambda_1, \ldots, \lambda_t \} \cup \{-\lambda_1, \ldots, -\lambda_t\}$ with $\lambda_i \neq \pm \lambda_j$ if $i \neq j$. By Corollary 3.4, the space $X^+ = \bigoplus_{i=1}^t V_{\lambda_i}$ is totally singular; hence, $\dim X^+ \leq n$ and

\[
\sum_{i=1}^t \# (V_{\lambda_i} \setminus \{0\}) \leq \# X^+ - 1 \leq q^n - 1;
\]

likewise, considering $X^- := \bigoplus_{i=1}^t V_{-\lambda_i}$, we get $\sum_{i=1}^t \# (V_{-\lambda_i} \setminus \{0\}) \leq q^n - 1$. Thus,

\[
A \geq q^{2n-2} - q^{2n-2r+1} - 2(q^n - 1).
\]

If $A > 0$, then $\text{wt}(\varphi) > q^{4n-5} - q^{2n-4}$. We now distinguish two cases.
Suppose that $\operatorname{Rad}(\varphi)$ contains a singular vector $u$; then, by statement (1) of Lemma 3.2
\[ X^+ \oplus \operatorname{Span}(u) \] would then be a totally singular subspace; thus, $\dim X^\pm \leq n - 1$ and
\[ A \geq q^{2n-2} - q^{2n-2r+1} - 2(q^{n-1} - 1) > q^{n-1}(q^{n-1} - q^{n-2} - 2) \geq 0; \]
therefore, $A > 0$. By Chevalley-Warning theorem for $2(n-r)+1 \geq 3$, the set $\operatorname{Rad}(\varphi) \cap Q$ always contains a non-zero singular vector.

Suppose now that $\operatorname{Rad}(\varphi)$ does not contain any non-zero singular vector; then $n = r$ and, consequently,
\[ A \geq q^{2n-2} - 1 - 2(q^n - 1) \] (where we have replaced by 1 the term $q^{2(n-r)+1}$ of (11),
which was an upper bound for the number of singular vectors in $\operatorname{Rad}(\varphi)$). This latter quantity is positive unless $n = 2$.

Therefore, $A \leq 0$ and $r > 1$ can occur only for $r = n = 2$.

If $A = 0$ then
\[ \#\{u: u \in Q \text{ and } u \text{ eigenvector of } T\} \cup \{0\} = q^2. \]
This happens only if there exists an eigenvalue $\lambda \neq 0$ such that $V_\lambda \subseteq Q$ and $\dim(V_\lambda) = 2$. By Lemma 3.3(4), also $-\lambda$ is an eigenvalue, so $V_{-\lambda} \subseteq Q$. Then
\[ \#\{u: u \in Q \text{ and } u \text{ eigenvector of } T\} \cup \{0\} > q^2, \text{ a contradiction.} \]

Hence $A < 0$ and $r > 1$. In this case $\operatorname{Rad}(\varphi)$ would be a one dimensional subspace of $V$ not contained in $Q$. We claim that actually $\varphi$ is in the $O(V^*)$-orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$. As before, let $\text{Spec}'(T) = \{\lambda_1, \ldots , \lambda_\ell\} \cup \{-\lambda_1, \ldots , -\lambda_\ell\}$. Since $X^+$ is totally singular, $\dim X \leq 2$, whence $\ell \leq 2$. If $\ell = 2$, then $\dim X^+ = \dim X^- = 2$. Thus, all four eigenspaces $V_{\pm \lambda_1}$ have dimension 1 and $\sum_{\lambda \in \text{Spec}'(T)} \#(V_\lambda \setminus \{0\}) = 4(q-1)$. It follows $A \geq q^2 - 1 - 4(q-1) = (q-2)^2 - 1 \geq 0$
and we are done. Therefore, $\ell \leq 1$. If $\ell = 0$, then $A \geq q^2 - 1 > 0$. Likewise, if $\ell = 1$ and $\dim V_{\lambda_1} = \dim V_{-\lambda_1} = 1$, then $A \geq q^2 - 1 - 2(q-1) > 0$. There remain to consider only the case $\ell = 1$ and $\dim V_\lambda = \dim V_{-\lambda} = 2$. Observe first that if there were a vector $b_3 \in V_{-\lambda} \cap V_+^Q$, then $V_\lambda \oplus \operatorname{Span}(b_3)$ would be totally singular — a contradiction, as the rank of $Q$ is 2. Therefore we can choose a basis $(b_1, b_2, \ldots , b_5)$ for $V$ such that $V_\lambda = \operatorname{Span}(b_1, b_3)$, $V_{-\lambda} = \operatorname{Span}(b_2, b_4)$, $\beta(b_2, b_1) = 1$, $\beta(b_3, b_2) = 0$, $\operatorname{Rad}(\varphi) = \operatorname{Span}(b_5)$ and $\operatorname{Span}(b_1, b_2, b_3)^Q = \operatorname{Span}(b_3, b_4)$. Indeed, we may assume that $b_3, b_4$ are a hyperbolic pair. By construction $\beta(Tb_4, b_1) = -\beta(b_4, Tb_1) = 0$
for $i = 1, 2, 4, 5$. Hence $T$ has matrix $\text{diag}(\lambda, -\lambda, \lambda, -\lambda, 0)$ with respect to this basis, that is $\varphi = b^1 \wedge b^2 + b^3 \wedge b^4$. We now compute $\operatorname{wt}(\varphi)$ directly, under the assumption $n = 2$ and obtain
\[ \operatorname{wt}(\varphi) = q^3 - q^2. \]
This completes the proof of the Main Theorem.

\[ \square \]

**Corollary 3.8**. If $n > 2$ the codewords of minimum weight all lie on the orbit of $e^1 \wedge e^{2n+1}$ under the action of the orthogonal group $O(V)$. For $n = 2$ the minimum weight codewords either lie in the orbit of $e^1 \wedge e^n$ or in the orbit of $e^1 \wedge e^2 + e^3 \wedge e^4$.

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