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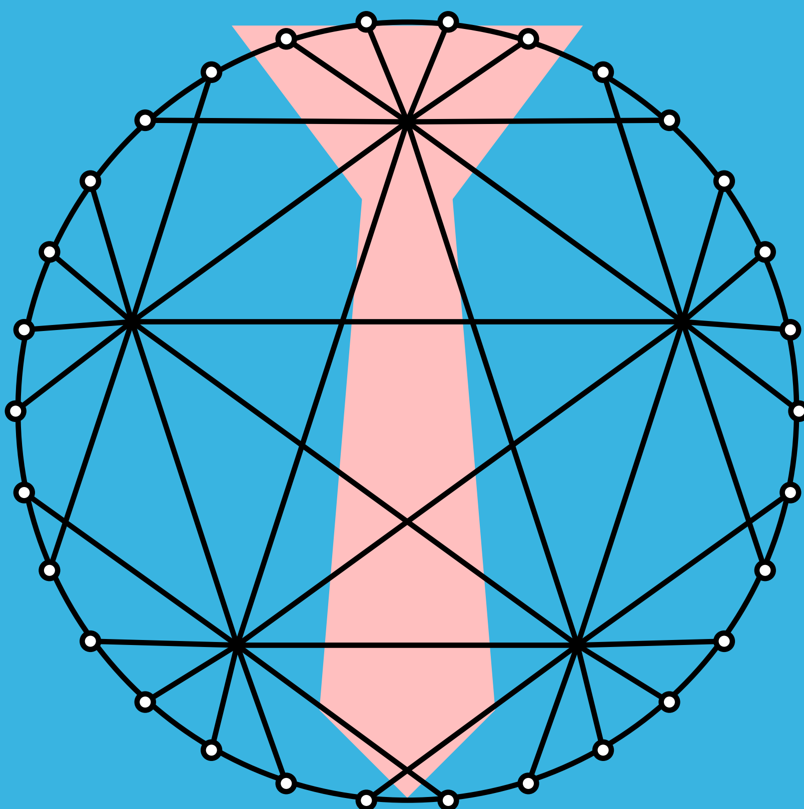

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Safe-sweep number of graphs

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Abstract. This paper presents an alternative edge-sweeping model where to account for unforeseen obstacles in a network, each cleaning device requires two units of battery to activate and loses one battery after each sweeping step. We study properties of this model and find bounds for the minimum number of batteries required to sweep all edges. In particular, we describe an algorithm that uses the minimum number of batteries when the network is a tree.

1 Introduction

Pretend you are responsible for the maintenance of roads in a city. In order to alleviate the workload, you put in place a network of battery-powered robots and battery charging stations at intersections and cul-de-sacs. These robots only need 1 battery to clean a road but you require them to depart with 2 batteries to clean a road in case of unforeseeable circumstances that would lengthen the travel time such as a large traffic jam. These robots go from one station to another, cleaning the road as they travel. If everything goes well, the robot leaves the additional battery at their arrival destination (charging station). Otherwise, the robot that encounters a hindrance uses the extra battery to finish cleaning the road that it is on and safely arrives to the charging station (instead of being stuck in the middle of the road) and alerts the city about the hindrance while maintenance of other roads is paused. Suppose you have sufficiently many robots at each station but overall a finite amount of batteries to supply for every station in the network. Assuming that robots face no hindrance before they complete the cleaning, what is the minimum total number of batteries you need to clean every road in the city and where do you place them?

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This current problem has elements in common with two well-studied graph processes known as graph brushing and graph pebbling. The brushing problem represents a decontamination scenario of the edges of a graph. In the initial version introduced by McKeil [7], one starts with a graph where all edges and vertices are initially dirty. Cleaning agents ("brushes") travel along incident dirty edges upon firing of the vertex, thereby cleaning the edges and the vertex itself. More than one brush is allowed to traverse an edge. The rule that allows a vertex to fire is that there are at least as many brushes on it as the number of incident dirty edges. For other versions of the brushing problem and more, see for example [2, 3, 8]. In the pebbling problem, initially there are some number of pebbles on some of the vertices of a graph, and these pebbles can be moved from a vertex to an adjacent vertex at a cost – one of the pebbles is lost when moving two pebbles from one vertex to an adjacent vertex. While graph pebbling is known to have been studied before, it owes its popularity to Chang [1]. The original version of the problem studies the minimum number n of pebbles needed so that any initial configuration of n pebbles on the vertices of a graph G allows for a series of pebbling moves to bring at least one pebble to any arbitrary target vertex – this minimum number of pebbles is called the pebbling number. This problem models a situation where any target vertex can be allocated resources after a sequence of pebbling moves, even when the fixed quantity of resources is initially distributed in a worst possible way. On the other hand, the optimal pebbling number is the smallest number n of pebbles needed so that there exists a distribution of n pebbles from which any target vertex can be reached by a sequence of pebbling moves. A series of papers by Hurlbert [4, 5, 6] surveys progress in graph pebbling, many of the variations of the problem and their applications.

In the current problem, vertices have a non-negative integer value representing the number of batteries at that station, and an edge can be cleaned only if one of the incident vertices has a value of at least 2. Then, the value of that vertex decreases by 2, and we increase the value of the other incident vertex by 1. An assignment of batteries to the vertices along with the order in which edges are cleaned following the aforementioned rule will be referred to as a "safe-sweeping" process $\mathcal{B}$; and if $\mathcal{B}$ cleans all the edges of the graph, then it is a complete safe-sweeping process. In this paper, for certain types of graphs we find algorithms that use the minimum number of batteries possible such that there exists a placement of those batteries that allows for a complete safe-sweeping process. We also give some basic bounds for graphs in general.

2 Some preliminaries

Let G be a graph without loops where each vertex is assigned a non-negative integer (number of "*batteries*"). A legal *sweeping* move of an edge $e = \{a, b\}$ reduces the number of batteries of one of the endpoints, say a, by 2 and increases the number of batteries of b by 1, in which case we say that *edge e is swept from a to b.* If the direction in which e is swept is irrelevant, then we may simply say that *edge e is swept.* Consequently, a legal sweeping of an edge $e = \{a, b\}$ can occur only if a or b have at least 2 batteries. Clean (swept) edges may be re-swept by the same rule.

We say $\mathcal{B}$ is a *safe-sweeping process* on a graph G if $\mathcal{B}$ describes an assignment of batteries to the vertices of G along with an ordering in which edges of G are swept by legal sweeping steps. If $\mathcal{B}$ sweeps all edges of G, then it is a *complete safe-sweeping process.* The example in Figure 1 illustrates a complete safe-sweeping process where at each step the vertex where the sweeping is initiated has the number of batteries given in a circle and swept edges are dashed.

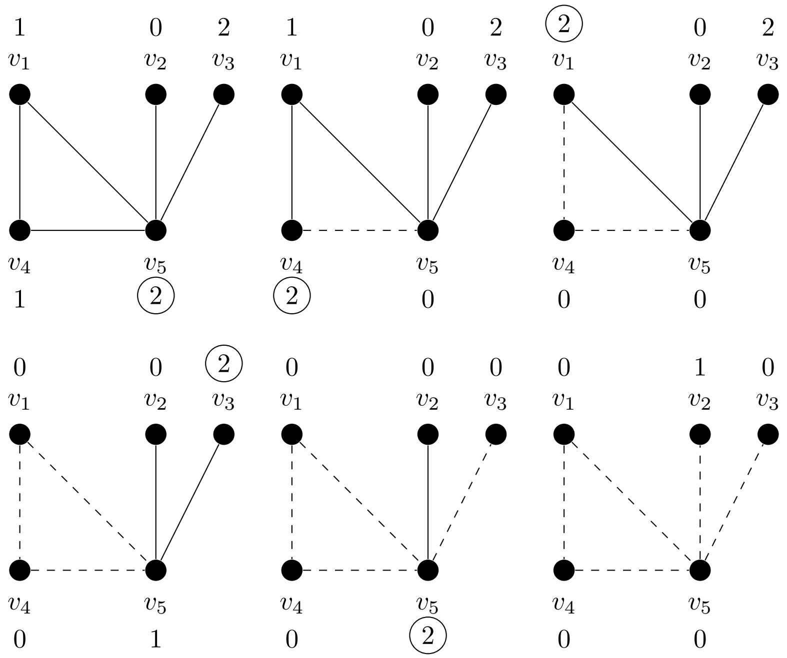

Figure 1: An example of a complete safe-sweeping process.

For a complete safe-sweeping process $\mathcal{B}$ of a graph G without loops, let $s_{\mathcal{B}}(G)$ denote the total number of batteries that were available on the vertices of G when $\mathcal{B}$ was initiated. We will denote by $ex_{\mathcal{B}}(G)$ the total number of excess batteries left at vertices of G after all edges are swept, and by $mult_{\mathcal{B}}(G)$ the total number of times clean edges are re-swept. Then, the following is a simple observation.

Lemma 2.1. *Let $\mathcal{B}$ be a complete safe-sweeping process of a non-empty graph G without loops. Then, $s_{\mathcal{B}}(G) = ex_{\mathcal{B}}(G) + mult_{\mathcal{B}}(G) + |E(G)|$.*

Proof. Each time an edge is swept, a battery is removed from the graph. Note that $|E(G)|$ and $mult_{\mathcal{B}}(G)$ account for all batteries that are removed in edge-sweepings while $ex_{\mathcal{B}}(G)$ accounts for the leftover batteries when $\mathcal{B}$ terminates. Hence $s_{\mathcal{B}}(G) = ex_{\mathcal{B}}(G) + mult_{\mathcal{B}}(G) + |E(G)|$. □

We define the *safe-sweep number* of a graph G without loops, denoted by $s(G)$, as $s(G) = \min\{s_{\mathcal{B}}(G) : \mathcal{B}$ is a complete safe-sweeping process of $G\}$ and we say $\mathcal{B}$ is an *optimal safe-sweeping process* of G if $s(G) = s_{\mathcal{B}}(G)$.

Let e be the last edge to be swept in a complete safe-sweeping process $\mathcal{B}$ of a connected graph G without loops. This last sweeping step results in one excess battery at one of the endpoints of e. Hence, by applying Lemma 2.1 to each connected component of G, we obtain the following simple bound.

Lemma 2.2. *For any non-empty graph G without loops that has k connected components, $s(G) \geqslant |E(G)| + k$.*

For graphs with Eulerian properties, an optimal safe-sweeping process is not difficult to find as highlighted in the next two results.

Theorem 2.3. *Let G be a non-empty graph without loops that has an Euler trail. Then, $s(G) = |E(G)| + 1$.*

Proof. Let v and w be the only odd-degree vertices in G. Place $\frac{deg(v)+3}{2}$ batteries at v, $\frac{deg(w)-1}{2}$ batteries at w, and $\frac{deg(z)}{2}$ batteries at any vertex $z \neq v, w$.

This placement of batteries ensures the existence a complete safe-sweeping process $\mathcal{B}$ along the Euler trail (that starts at v and finishes at w) since each vertex will have at least 2 batteries every time it is visited along the

Euler trail. Note that when this complete safe-sweeping process terminates, exactly 1 battery will remain at w.

It is easy to see that $\mathcal{B}$ uses exactly $|E(G)|+1$ batteries; hence, by Lemma 2.2, we obtain

$$s(G) = |E(G)| + 1.$$

□

Corollary 2.4. *Let G be a non-empty graph without loops that has an Euler tour. Then, $s(G) = |E(G)| + 1$. Additionally, for any vertex $v \in V(G)$, there exists a complete safe-sweeping process $\mathcal{B}$ with the following two properties:*

(1) $s_{\mathcal{B}}(G) = s(G) = |E(G)| + 1$, *and*

(2) *the unique excess battery at the end of the safe-sweeping process is located at v.*

Before stating and proving a critical lemma next, we shall first introduce some terminology.

Let u be an arbitrary vertex in a non-empty graph G without loops and $\mathcal{B}$ be an arbitrary complete safe-sweeping process on G such that u has at least one excess battery (that was not initially at u) when $\mathcal{B}$ terminates. We will define a *battery path* of u in $\mathcal{B}$ as a vertex sequence of maximal length $(v_1, v_2, v_3, \ldots, v_k = u)$

(1) 2 or more batteries were initially placed in $\mathcal{B}$ at v_1,

(2) v_i is adjacent to v_{i+1} in G for each $i \in \{1, 2, \ldots, k-1\}$, and

(3) $\mathcal{B}$ sweeps the edges induced by the vertex sequence in the order $v_1, v_2, \ldots, v_k$ from v_i to v_{i+1} for all $i \in \{1, 2, \ldots, k-1\}$. (Note that these edge-sweepings do not necessarily happen at consecutive steps in $\mathcal{B}$.)

Battery paths give us a way to follow the sweepings of edges and movement of batteries that result in some excess battery. For example, in Figure 1 the only excess battery is located at v_2 when $\mathcal{B}$ terminates. A battery path of v_2 is (v_3, v_5, v_2), another one is $(v_5, v_4, v_1, v_5, v_2)$. More examples of battery paths are found in Examples 2.5 and 2.7.

We say that an edge $e = \{v, w\}$ is swept *inwards* to vertex v if 1 battery was added to v when the sweeping of e was complete. Similarly, we say that an edge e is swept *outwards* from vertex v if 2 batteries were removed from v when e was swept. We may use $in_{\mathcal{B}}(v)$ and $out_{\mathcal{B}}(v)$ to denote the number of edge-sweepings inwards to v and outwards from v, respectively, in a complete safe-sweeping process $\mathcal{B}$. If no edges are re-swept in $\mathcal{B}$, then $in_{\mathcal{B}}(v)$ and $out_{\mathcal{B}}(v)$ simply refer to the number of distinct edges incident with v that are swept inwards to v and outwards from v, respectively. Sweeping inwards or outwards denotes the *direction* in which an edge is swept with respect to a vertex v. Considering the direction in which a given edge is swept with respect to a specified incident vertex proves to be useful for the remainder of this paper. The following observation is helpful for the reversibility theorem that we will prove next.

Observation: Let $\mathcal{B}$ be a complete safe-sweeping process on a graph G without loops. Let $\mathcal{E}_{\mathcal{B}} = (e_1, e_2, \ldots, e_t)$ be the ordering in which edges are swept in $\mathcal{B}$. Let v be a vertex of G that has at least one excess battery (that was not initially at v) when $\mathcal{B}$ terminates. Consider a battery path P of v in $\mathcal{B}$, and let $\mathcal{E}_{\mathcal{B},P}$ be an *ordering of edge-sweepings associated to P in $\mathcal{B}$* that is formed as follows:

Step 0: Include the first edge of P in $\mathcal{E}_{\mathcal{B},P}$.

Step 1: If $\mathcal{E}_{\mathcal{B},P}$ currently allows for a legal sweeping of the next edge of P, then add the next edge of P to $\mathcal{E}_{\mathcal{B},P}$. Keep adding edges of P to $\mathcal{E}_{\mathcal{B},P}$ until $\mathcal{E}_{\mathcal{B},P}$ does not allow for a legal sweeping of the next edge of P.

Step 2: Let the last added edge of P be $e_j = \{s, t\}$, which is swept from s to t in $\mathcal{B}$. (In this case, we will refer to t as the *crossing vertex* of P.) Then, there must be some other battery path of v that contains t such that the subpath P' of that battery path that stops at t consists of unswept edges. Add edges of P' to $\mathcal{E}_{\mathcal{B},P}$ in order until $\mathcal{E}_{\mathcal{B},P}$ does not allow for a legal sweeping of the next edge of P'.

Step 3: If all edges of P' have been added to $\mathcal{E}_{\mathcal{B},P}$, we return to Step 1 and add the next edge of P to $\mathcal{E}_{\mathcal{B},P}$. Otherwise, we return to Step 2 and apply it on P' and its crossing vertex.

The procedure stops when all edges of P are included in $\mathcal{E}_{\mathcal{B},P}$.
Then, $\mathcal{B}'$ with the same battery placement and sweeping direction of edges as in $\mathcal{B}$ also determines a complete safe-sweeping process on G if

(i) $\mathcal{B}'$ respects the relative ordering of edge-sweepings in $\mathcal{B}$ of edges not in $\mathcal{E}_{\mathcal{B},P}$, and

(ii) $\mathcal{B}'$ respects the relative ordering of edge-sweepings in $\mathcal{B}$ of edges in $\mathcal{E}_{\mathcal{B},P}$, or $\mathcal{E}_{\mathcal{B},P}$ is a subsequence of $\mathcal{E}_{\mathcal{B}'}$.

This observation is illustrated in the next example.

Example 2.5. In Figure 2, let $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}$, $e_4 = \{v_4, v_5\}, e_5 = \{v, v_4\}, e_6 = \{v_8, v_9\}, e_7 = \{v_8, v_{10}\}, e_8 = \{v_8, v_{11}\}, e_9 = \{v_8, v_{12}\}, e_{10} = \{v_6, v_8\}, e_{11} = \{v_7, v_8\}, e_{12} = \{v_6, v_7\}, e_{13} = \{v, v_6\}$, and consider the complete safe-sweeping process $\mathcal{B}$ with the ordering of the edge-sweepings given as $\mathcal{E}_{\mathcal{B}} = (e_1, e_2, \ldots, e_{13})$. Note that v has an excess battery (that was not initially at v) when $\mathcal{B}$ terminates. A battery path of v is $P = (v_{10}, v_8, v_7, v_6, v)$. Here, the edges of P are $e_7, e_{11}, e_{12}, e_{13}$ and they are swept in that order in $\mathcal{B}$.

We form an ordering $\mathcal{E}_{\mathcal{B},P}$ of edge-sweepings associated to P in $\mathcal{B}$ as follows: Add e_7 to $\mathcal{E}_{\mathcal{B},P}$. Currently $\mathcal{E}_{\mathcal{B},P}$ does not allow for a legal sweeping of e_{11}; hence, v_8 is a crossing vertex. Consider the battery path (v_9, v_8, v_7, v_6, v) of v, where $P' = (v_9, v_8)$ is a subpath that stops at v_8 and consists of unswept edges only. Add e_6 to $\mathcal{E}_{\mathcal{B},P}$. Now we may sweep e_{11} and then e_{12} on P; so, add e_{11} and e_{12} to $\mathcal{E}_{\mathcal{B},P}$. Currently $\mathcal{E}_{\mathcal{B},P}$ does not allow for a legal sweeping of e_{13}; hence, v_6 is a crossing vertex. Consider the battery path (v_{11}, v_8, v_6, v) of v, where $P' = (v_{11}, v_8, v_6)$ is a subpath that stops at v_6 and consists of unswept edges only. Add e_8 to $\mathcal{E}_{\mathcal{B},P}$. Note that currently $\mathcal{E}_{\mathcal{B},P}$ does not allow for a legal sweeping of e_{10}; hence, v_8 is a crossing vertex. Consider the battery path (v_{12}, v_8, v_6, v) of v, where $P' = (v_{12}, v_8)$ is a subpath that stops at v_8 and consists of unswept edges only. Add e_9 to $\mathcal{E}_{\mathcal{B},P}$. Now we may sweep e_{10} and then e_{13} on P; so, add e_{10} and e_{13} to $\mathcal{E}_{\mathcal{B},P}$. All edges of P are included in $\mathcal{E}_{\mathcal{B},P}$; so, the procedure stops resulting in $\mathcal{E}_{\mathcal{B},P} = (e_7, e_6, e_{11}, e_{12}, e_8, e_9, e_{10}, e_{13})$.

Then, for example, $\mathcal{B}'$ with the same battery placement and sweeping direction of edges as in $\mathcal{B}$, where

$$\mathcal{E}_{\mathcal{B}'} = (e_1, e_2, e_7, e_3, e_6, e_4, e_{11}, e_{12}, e_8, e_5, e_9, e_{10}, e_{13})$$

also determines a complete safe-sweeping process. Note that $\mathcal{B}'$ respects the relative ordering of edge-sweepings in $\mathcal{B}$ of edges not in $\mathcal{E}_{\mathcal{B},P}$ and $\mathcal{E}_{\mathcal{B},P}$ is a subsequence of $\mathcal{E}_{\mathcal{B}'}$. Similarly,

$$\mathcal{E}_{\mathcal{B}''} = (e_1, e_2, e_6, e_7, e_3, e_4, e_8, e_9, e_{10}, e_5, e_{11}, e_{12}, e_{13})$$

also determines a complete safe-sweeping process. Note that $\mathcal{B}''$ respects the relative ordering of edge-sweepings in $\mathcal{B}$ of edges not in $\mathcal{E}_{\mathcal{B},P}$ and it respects the relative ordering of edge-sweepings in $\mathcal{B}$ of edges that are in $\mathcal{E}_{\mathcal{B},P}$.

Lemma 2.6. (Correction Algorithm) *Let G be a non-empty graph without loops and $\mathcal{B}$ be an optimal complete safe-sweeping process of G. When $\mathcal{B}$ terminates, no vertex in G has 2 or more excess batteries.*

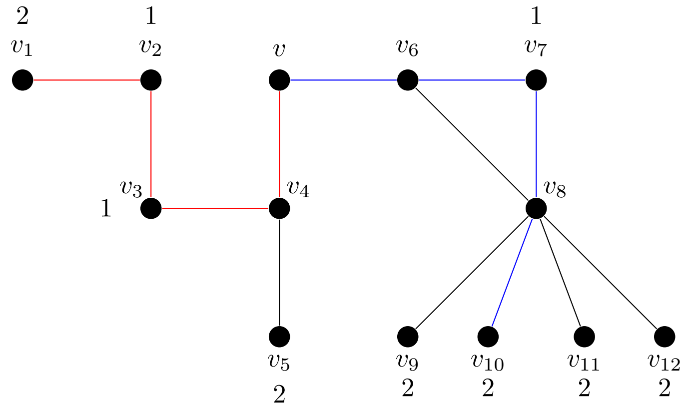

Figure 2: Examples of battery paths.

Proof. Let G be a non-empty graph without loops, $\mathcal{B}$ be an optimal complete safe-sweeping process of G, and let v be a vertex in G that has 2 or more batteries after $\mathcal{B}$ terminates. We will show that the existence of v contradicts the optimality of $\mathcal{B}$. We shall do so by considering the two cases as below.

Case 1: Suppose that at most one inwards edge-sweeping occurred at v in $\mathcal{B}$. Since v has two or more batteries when $\mathcal{B}$ terminates, it must be that at least one of those batteries was initially placed at v in $\mathcal{B}$. Then, we can reduce the number of batteries initially placed at v in $\mathcal{B}$ by one, and the resulting assignment of batteries can still sweep all edges of G in the same way as in $\mathcal{B}$. This contradicts the assumption that $\mathcal{B}$ is optimal.

Case 2: Suppose that at least two inwards edge-sweepings occurred at v in $\mathcal{B}$. Then (at least) two batteries were brought to v through edge-sweepings when $\mathcal{B}$ terminated. Consider two battery paths of v in $\mathcal{B}$, say $(x_1, \ldots, x_j = v)$ and $(y_1, \ldots, y_k = v)$ chosen in such a way that each battery path corresponds to the movement of a distinct battery in $\mathcal{B}$ towards v. (Note that these battery paths are not necessarily disjoint.)

We alter the complete safe-sweeping process $\mathcal{B}$ to obtain a safe-sweeping process $\mathcal{B}'$ by initially placing 1 additional battery at v and removing 2 batteries from y_1. We impose that $\mathcal{B}'$ sweeps all edges in the same order and direction as it was in $\mathcal{B}$, except for those that are on the path $(y_1, \ldots, y_k = v)$. This results in v having at least 2 batteries (because of the additional battery we placed at v), which now allows the path $(y_1, \ldots, y_k = v)$ to be swept in the opposite order and direction than it was in $\mathcal{B}$. Observe that $\mathcal{B}'$ is a complete safe-sweeping process of G with one fewer battery than in $\mathcal{B}$ – a contradiction to the optimality of $\mathcal{B}$. $\square$

The next example illustrates Case 2 of the Correction Algorithm.

Example 2.7. The graph in Figure 2 along with the initial placement of batteries possesses a complete safe-sweeping process $\mathcal{B}$ that terminates leaving two batteries at vertex v. For example, one can sweep the edges in the following order: $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_5, v_4\}, \{v_4, v\}, \{v_9, v_8\}, \{v_{10}, v_8\}, \{v_8, v_7\}, \{v_7, v_6\}, \{v_{11}, v_8\}, \{v_{12}, v_8\}, \{v_8, v_6\}, \{v_6, v\}$. The sweeping of $\{v_4, v\}$ brings one battery (say, battery 1) to v and the sweeping of $\{v_6, v\}$ brings another battery (say, battery 2) to v.

There are multiple choices for battery paths of v in $\mathcal{B}$ corresponding to the two batteries. For example, a battery path of v in $\mathcal{B}$ corresponding to battery 1 is (v_1, v_2, v_3, v_4, v) highlighted in red in Figure 2; another one is (v_5, v_4, v). A battery path of v in $\mathcal{B}$ corresponding to battery 2 is $(v_{10}, v_8, v_7, v_6, v)$ highlighted in blue in Figure 2; another one is (v_{12}, v_8, v_6, v).

To apply the Correction Algorithm, we may consider battery paths

$$(v_1, v_2, v_3, v_4, v)(\text{for battery } 1)$$

and

$$(v_{10}, v_8, v_7, v_6, v)(\text{for battery } 2)$$

of v in $\mathcal{B}$. We form a new complete safe-sweeping process $\mathcal{B}'$ from $\mathcal{B}$ as in Figure 3. The order of edge-sweepings in $\mathcal{B}'$ is $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_5, v_4\}$, $\{v_4, v\}$, $\{v_9, v_8\}$, $\{v_{11}, v_8\}$, $\{v_{12}, v_8\}$, $\{v_8, v_6\}$, $\{v, v_6\}$, $\{v_6, v_7\}$, $\{v_7, v_8\}$, $\{v_8, v_{10}\}$.

Lemma 2.8. *Let G be a non-empty graph without loops. There is a complete safe-sweeping process of G with $s(G)$ batteries that does not re-sweep any edges of G.*

Proof. Let G be a non-empty graph without loops, and let $\mathcal{B}$ be an optimal complete safe-sweeping process of G such that $\mathcal{B}$ sweeps some edge $e = \{u, v\}$ in G at least twice. Without loss of generality, assume that the last sweeping of e occurs from u to v.

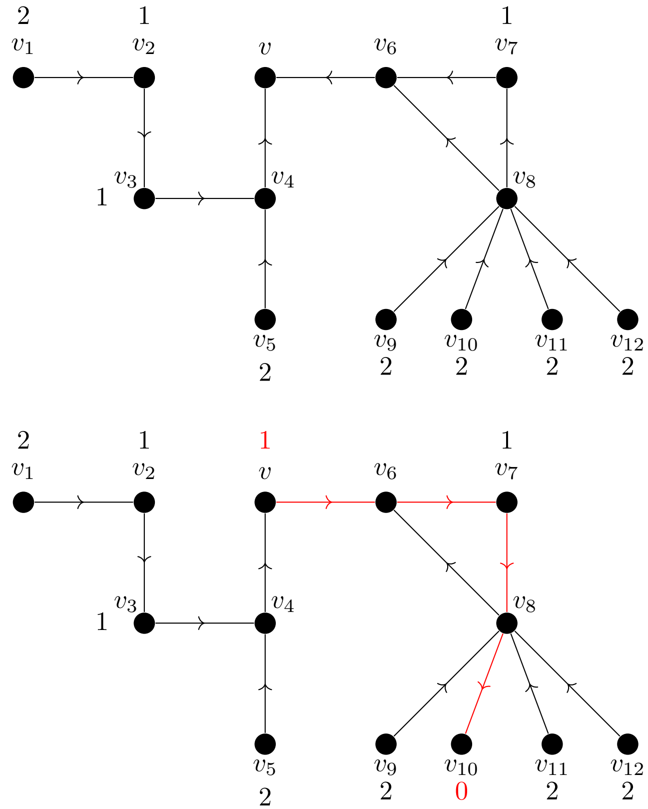

Figure 3: The direction in which edges are swept in $\mathcal{B}$ and $\mathcal{B}'$.

Let $\mathcal{B}'$ be a safe-sweeping process of G that sweeps edges of G in the same direction and order as $\mathcal{B}$ up until $\mathcal{B}$ sweeps e for the last time. In order for $\mathcal{B}$ to sweep e, u must currently have at least 2 batteries. In $\mathcal{B}'$, instead of sweeping e again, we shall place one battery at v and not sweep e. Since v has now been compensated for the battery that it would have received from sweeping e again in $\mathcal{B}$, $\mathcal{B}'$ can proceed sweeping the remaining edges in the same direction and order as $\mathcal{B}$. Observe that $\mathcal{B}'$ is a complete safe-sweeping process which uses one more battery than $\mathcal{B}$.

Since the batteries (at least 2) accumulated at u to sweep e for the last time in $\mathcal{B}$ were not used in $\mathcal{B}'$, when $\mathcal{B}'$ terminates, u will have at least 2 excess batteries. Then, by the Correction Algorithm as was described in the proof of Lemma 2.6, there exists a complete safe-sweeping process $\mathcal{B}''$ of

G which uses one less battery than $\mathcal{B}'$. Moreover, $\mathcal{B}''$ does not sweep e more than $\mathcal{B}'$ does because the Correction Algorithm only alters the number of batteries for certain vertices or changes the direction in which edges are swept. Hence, $\mathcal{B}''$ uses the same number of batteries as $\mathcal{B}$ and sweeps e one less time than $\mathcal{B}$ does. We can repeat this procedure in $\mathcal{B}$ for all re-sweepings of edges until we obtain a complete safe-sweeping process of G which does not re-sweep any edges while still only using $s(G)$ batteries. □

For the remainder of this paper, in light of Lemma 2.8, we shall only consider optimal complete safe-sweeping processes which do not re-sweep any edges unless stated otherwise.

Lemma 2.9. (Reversibility) *Let $\mathcal{B}$ be a complete safe-sweeping process on a graph G without loops such that no edges are re-swept in $\mathcal{B}$. Suppose $v \in V(G)$ has an excess battery (that was not initially at v) at the end of process $\mathcal{B}$. Consider a battery path P of v. Remove two batteries from the first vertex of P and place two batteries at the last vertex of P. Then, there is a complete safe-sweeping process $\mathcal{B}^{-1}$ on G with the new battery assignment such that $s_{\mathcal{B}^{-1}}(G) = s_{\mathcal{B}}(G)$.*

Proof. Let $\mathcal{B}, G$ and P be defined as above. Let $\mathcal{E}_{\mathcal{B}} = (e_1, e_2, \ldots, e_t)$ be the ordering of edge-sweepings in $\mathcal{B}$. Let $\mathcal{E}_{\mathcal{B},P}$ be an ordering of edge-sweepings associated to P in $\mathcal{B}$ as defined in the observation preceding Example 2.5. In light of the observation, without loss of generality we may suppose that the edges in $\mathcal{E}_{\mathcal{B},P}$ are $e_{s_i}, e_{s_i+1}, \ldots, e_t$ (so that all such edges are located at the end of $\mathcal{E}_{\mathcal{B}}$). Remove two batteries from the first vertex of P, place two batteries at the last vertex of P and form $\mathcal{E}^{-1}_{\mathcal{B},P} = (e_1, e_2, \ldots, e_{s_i-1}, e'_{s_i}, e'_{s_i+1}, \ldots, e'_{t-1}, e'_t)$ where $e'_{s_i}, \ldots, e'_t$ are the edges in $\mathcal{E}_{\mathcal{B},P}$ ordered in a way that edges not belonging to P preserve their relative ordering in $\mathcal{E}_{\mathcal{B}}$ and occupy the positions immediately after e_{s_i-1} in $\mathcal{E}^{-1}_{\mathcal{B},P}$, while edges belonging to P are at the end of $\mathcal{E}^{-1}_{\mathcal{B},P}$ and in a reverse order (so, e'_t is the first edge on P, e'_{t-1} is the second edge on P and so on). Then, $\mathcal{E}^{-1}_{\mathcal{B},P}$ determines a complete safe-sweeping process $\mathcal{B}^{-1}$ such that $s_{\mathcal{B}^{-1}}(G) = s_{\mathcal{B}}(G)$ where the edges on P are swept in the reverse direction as opposed to in $\mathcal{B}$. □

Proposition 2.10. *Let G be a non-empty graph without loops, and let $e \in E(G)$. Then, $s(G) \geqslant s(G - e)$.*

Proof. Let G be a non-empty graph without loops and $\mathcal{B}$ be an optimal complete safe-sweeping process of G. Without loss of generality, suppose that $e = \{u, v\}$ is swept from u to v in $\mathcal{B}$. Consider a safe-sweeping process $\mathcal{B}'$ of $G - e$ which places an additional battery at v and sweeps edges in the same direction and order as $\mathcal{B}$ for all edges other than e. Note that v has been compensated for the battery it would have received in $\mathcal{B}$ from the sweeping of e. Then, $\mathcal{B}'$ is a complete safe-sweeping process of $G - e$ which uses $s(G) + 1$ batteries. Moreover, when $\mathcal{B}'$ terminates, u has at least 2 batteries (that were originally used to sweep e in $\mathcal{B}$). Then, by the Correction Algorithm $\mathcal{B}'$ can be altered to use at least one less battery to obtain a complete safe-sweeping process of $G - e$ which uses at most $s(G)$ batteries. □

Lemma 2.11. (Degree 2 Contraction) *Let G be a non-empty graph and $v \in V(G)$ be a vertex of degree 2 that is adjacent to distinct vertices $u \neq v$ and $w \neq v$ in G. Consider the graph G' obtained from G by removing v and adding an edge $\{u, w\}$. Then, $s(G') + 1 = s(G)$.*

Proof. First, we will show that $s(G') + 1 \leqslant s(G)$. Let $\mathcal{B}$ be an optimal complete safe-sweeping process of G. By Lemma 2.8, we may assume that $\mathcal{B}$ does not re-sweep any edges of G. Note that $\mathcal{B}$ sweeps the two edges incident with v in one of the following two ways.

(1) Unidirectionally, i.e., from u to v and then from v to w (or from w to v and then from v to u), or
(2) Outwards only, i.e., from v to u and then from v to w.

Note that we do not need to consider the case where both edges incident with v are swept inwards since that would result in v having two excess batteries when $\mathcal{B}$ terminates, a contradiction to the optimality of $\mathcal{B}$ by Lemma 2.6. Similarly, by the optimality of $\mathcal{B}$, we do not need to consider the case where one of the two edges incident with v is swept away from v before the other edge is swept towards v.

Case 1: (Unidirectionally) Without loss of generality, suppose that the sweepings of the edges incident with v are from u to v and then from v to w. Since no edges of G are re-swept in $\mathcal{B}$, there must be one battery initially placed at v in $\mathcal{B}$ so that $\mathcal{B}$ can sweep from v to w.

Let $\mathcal{B}'$ be a safe-sweeping process of G' which follows the same battery assignment and the same sweeping steps as in $\mathcal{B}$ for edges that G and G' have in common until $\mathcal{B}$ sweeps the edge $\{v, w\}$; and when $\mathcal{B}$ sweeps the edge $\{v, w\}$ in G, $\mathcal{B}'$ shall sweep the edge $\{u, w\}$ from u to w. This allows $\mathcal{B}'$ to continue sweeping edges in the same direction and order as in $\mathcal{B}$

since the battery that w would have received from the sweeping of $\{v, w\}$ in $\mathcal{B}$ has been compensated in $\mathcal{B}'$ by the battery received at w from the sweeping of the edge $\{u, w\}$. Therefore, $\mathcal{B}'$ is a complete safe-sweeping process; and because v is not present in G', $\mathcal{B}'$ uses at most $s(G) - 1$ batteries. Hence, $s(G') + 1 \leqslant s(G)$.

Case 2: (Outwards only) Without loss of generality, suppose that the edge $\{u, v\}$ is swept in an earlier step than the edge $\{v, w\}$ in $\mathcal{B}$. Since no edges of G are re-swept in $\mathcal{B}$, there must be 4 or more batteries initially placed at v in $\mathcal{B}$ so that $\mathcal{B}$ can sweep edges $\{u, v\}$ and $\{v, w\}$.

Let $\mathcal{B}'$ be a safe-sweeping process of G' which has 3 more batteries initially placed at w than in $\mathcal{B}$, and follows the same battery assignment elsewhere along with the same sweeping steps as in $\mathcal{B}$ for edges that G and G' have in common until $\mathcal{B}$ sweeps $\{u, v\}$. At the step when $\mathcal{B}$ sweeps $\{u, v\}$, $\mathcal{B}'$ shall sweep the edge $\{u, w\}$ from w to u. This adjustment ensures that after this step, u and w have at least as many batteries in $\mathcal{B}'$ as they would in $\mathcal{B}$ after the sweeping of $\{u, v\}$. Note that this is true before and after the sweeping of edge $\{v, w\}$ in $\mathcal{B}$. Therefore, $\mathcal{B}'$ can continue sweeping edges in the same direction and order as in $\mathcal{B}$. Consequently, $\mathcal{B}'$ is a complete safe-sweeping process; and because v is not present in G', $\mathcal{B}'$ uses at most $s(G) - 1$ batteries. Hence, $s(G') + 1 \leqslant s(G)$.

Next, we will show that $s(G) \leqslant s(G') + 1$. Let $\mathcal{B}'$ be an optimal complete safe-sweeping process of G', and without loss of generality suppose that edge $e = \{u, w\}$ in G' is swept from u to w in $\mathcal{B}'$. Let $\mathcal{B}$ be a safe-sweeping process of G which assigns the same number of batteries as $\mathcal{B}'$ does to the vertices that G and G' have in common; additionally, $\mathcal{B}$ places 1 battery at v. We impose that $\mathcal{B}$ follows the same edge-sweeping steps as in $\mathcal{B}'$ with the only difference that when $\mathcal{B}'$ sweeps e in G', $\mathcal{B}$ shall sweep the edges incident with v in G unidirectionally from u to v and then from v to w. Observe that $\mathcal{B}$ is a complete safe-sweeping process of G which uses exactly 1 more battery than $\mathcal{B}'$; consequently, we have $s(G) \leqslant s(G') + 1$.

We have shown $s(G') + 1 \leqslant s(G)$ and $s(G) \leqslant s(G') + 1$; hence

$$s(G) = s(G') + 1. \qquad \square$$

Lemma 2.11 implies that the presence of each vertex of degree 2 in a graph (other than possibly those on components that are loops or multiple edges) increases the safe-sweep number of the associated graph by exactly 1. Therefore, to find $s(G)$ of a given graph G, we may apply Lemma 2.11 until we obtain a graph G' which has no vertices of degree 2 (other than possibly those on components that are loops or multiple edges), and study $s(G')$ instead.

3 Main results

In this section, we present our main result (Theorem 3.4) which determines the safe-sweep number of trees, along with an algorithm for finding an optimal complete safe-sweeping process for any given tree. Before we state and prove our main result, first we determine the safe-sweep number of stars.

Theorem 3.1. *Let $K_{1,k}$ be the star with $k \geqslant 1$ edges. Then, $s(K_{1,k}) = \left\lceil \frac{4k}{3} \right\rceil$.*

Proof. Let $\mathcal{B}$ be an optimal complete safe-sweeping process of $K_{1,k}$. By Lemma 2.8, we may assume that $\mathcal{B}$ does not re-sweep edges. Note that all edges in $K_{1,k}$ are pendant edges. In $\mathcal{B}$, we will denote by $out_{\mathcal{B}}(K_{1,k})$ the number of edges in $K_{1,k}$ that are swept outwards from the central vertex. Similarly, we will denote by $in_{\mathcal{B}}(K_{1,k})$ the number of edges in $K_{1,k}$ that are swept inwards to the central vertex.

Each time $\mathcal{B}$ sweeps an edge e of $K_{1,k}$ outwards, the number of batteries at the central vertex goes down by 2, while 1 battery is added to the corresponding leaf incident with e. Since $\mathcal{B}$ does not re-sweep edges, such a battery never moves again in $\mathcal{B}$, and it contributes 1 to $ex_{\mathcal{B}}(K_{1,k})$. Therefore, by Lemma 2.1,

$$s_{\mathcal{B}}(K_{1,k}) \geqslant |E(K_{1,k})| + out_{\mathcal{B}}(K_{1,k}) = k + out_{\mathcal{B}}(K_{1,k}). \tag{I}$$

Note that $out_{\mathcal{B}}(K_{1,k}) = k - in_{\mathcal{B}}(K_{1,k})$. By substituting this identity into (I), we obtain

$$s_{\mathcal{B}}(K_{1,k}) \geqslant 2k - in_{\mathcal{B}}(K_{1,k}). \tag{II}$$

Observe that any edge e of $K_{1,k}$ to be swept inwards requires two batteries to be initially placed in $\mathcal{B}$ at the leaf incident with e. Therefore,

$$s_{\mathcal{B}}(K_{1,k}) \geqslant 2 \cdot in_{\mathcal{B}}(K_{1,k}). \tag{III}$$

From (II) and (III), we obtain

$$s(K_{1,k}) = s_{\mathcal{B}}(K_{1,k}) \geqslant \max\{2k - in_{\mathcal{B}}(K_{1,k}), 2 \cdot in_{\mathcal{B}}(K_{1,k})\}. \tag{IV}$$

Routine computations reveal that the minimum value of the lower bound in ((IV)) is obtained when $in_{\mathcal{B}}(K_{1,k}) = \left\lfloor \frac{2k}{3} \right\rfloor$, yielding the minimum value $\left\lceil \frac{4k}{3} \right\rceil$. Hence, we obtain

$$s(K_{1,k}) \geqslant \left\lceil \frac{4k}{3} \right\rceil. \tag{V}$$

In the remainder of the proof, for each positive integer k, we describe a complete safe-sweeping process $\mathcal{B}$ of $K_{1,k}$ that uses exactly $\left\lceil \frac{4k}{3} \right\rceil$ batteries. This is done in three parts, by considering $k \equiv 0 \pmod 3$, $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$ in turn.

$k \equiv 0 \pmod 3$: Consider three edges of $K_{1,k}$. Choose two of them to be swept inwards. This allows the last edge to be swept outwards with no additional battery. Four batteries are used to sweep three edges. Thus, $\mathcal{B}$ requires $\frac{4k}{3}$ batteries in total, which is equal to $\left\lceil \frac{4k}{3} \right\rceil$ since $k \equiv 0 \pmod 3$.

$k \equiv 1 \pmod 3$: All but one edge of $K_{1,k}$ may be bundled into triples and swept accordingly as was described in the previous case. We may sweep the remaining edge outwards by placing two batteries initially in $\mathcal{B}$ at the corresponding leaf. Therefore, $\mathcal{B}$ requires $\frac{4(k-1)}{3} + 2$ batteries, which is equal to $\left\lceil \frac{4k}{3} \right\rceil$, since $k \equiv 1 \pmod 3$.

$k \equiv 2 \pmod 3$: All but two edges of $K_{1,k}$ may be bundled into triples and swept accordingly as was described in the first case. To sweep the remaining two edges, we may initially place in $\mathcal{B}$ one battery at the central vertex and two batteries at the leaf incident with one of these two edges. Sweep the edge with two batteries at the corresponding leaf inwards (towards the central vertex). Now the central vertex has two batteries which allows $\mathcal{B}$ to sweep the other edge outwards. Note that $\mathcal{B}$ uses $\frac{4(k-2)}{3} + 3$ batteries, which is equal to $\left\lceil \frac{4k}{3} \right\rceil$, since $k \equiv 2 \pmod 3$.

For all $k \geqslant 1$ we have described a complete safe-sweeping process of $K_{1,k}$ that uses exactly $\left\lceil \frac{4k}{3} \right\rceil$ batteries, thereby finishing the proof. □

A tree is said to be *starlike* if it has exactly one vertex of degree greater than 2. Theorem 3.1 can be further generalized to starlike trees.

Corollary 3.2. *Suppose that a starlike tree S has p vertices of degree 2. Then, $s(S) = \left\lceil \frac{4m}{3} \right\rceil + p$ where m is the degree of the only vertex in S with degree greater than 2.*

Proof. By repeatedly applying Lemma 2.11 to S until all p vertices of degree 2 disappear, we obtain a star with m edges. Then, by Lemma 2.11 we have $s(S) = s(K_{1,m}) + p$, which is equal to $\left\lceil \frac{4m}{3} \right\rceil + p$ by Theorem 3.1. □

Observation (⋆): Let $\mathcal{B}$ be a complete safe-sweeping process on a graph G without loops such that no edges are re-swept in $\mathcal{B}$. For any $v \in V(G)$, if v has no batteries at both the beginning and end of process $\mathcal{B}$, then $\deg(v) \equiv 0 \pmod 3$.

Proof of Observation: Let $\mathcal{B}$ and G be as described above, and suppose that $v \in V(G)$ has no batteries at both the beginning and end of process $\mathcal{B}$. Since v has no batteries at the beginning of $\mathcal{B}$, $in_{\mathcal{B}}(v) \geqslant 2 \cdot out_{\mathcal{B}}(v)$. Suppose that $in_{\mathcal{B}}(v) > 2 \cdot out_{\mathcal{B}}(v)$. Then, v will have batteries remaining at the end of process $\mathcal{B}$, which is a contradiction. Therefore, $in_{\mathcal{B}}(v) = 2 \cdot out_{\mathcal{B}}(v)$. Consequently, since no edges are re-swept in $\mathcal{B}$, $\deg(v) \equiv 0 \pmod 3$. □

The next theorem is the key result towards determining the safe-sweep number of trees.

Theorem 3.3. *Let G be a graph without loops. Suppose that there is $u \in V(G)$ such that $\deg_G(u) = 1$. Let $G' = G - u$ and w be the unique neighbour of u in G. Then,*

$$s(G) = \begin{cases} s(G') + 1 & \text{if } \deg_{G'}(w) \not\equiv 0 \pmod 3 \\ s(G') + 2 & \text{if } \deg_{G'}(w) \equiv 0 \pmod 3. \end{cases}$$

Proof. Let G be a graph without loops. Suppose that there is $u \in V(G)$ such that $\deg_G(u) = 1$, and let w be the unique neighbour of u in G. Let $G' = G - u$. Consider an optimal complete safe-sweeping process $\mathcal{B}$ on G that does not re-sweep edges. Since $\deg_G(u) = 1$, either u initially has 2 batteries (edge $\{u, w\}$ is swept from u to w) or has 1 battery at the end of the process (edge $\{u, w\}$ is swept from w to u).

In the former case, move a battery from u to w and delete u along with the battery remaining at it to create $\mathcal{B}'$ where apart from the sweeping of the edge $\{u, w\}$, the sweeping ordering is the same as in $\mathcal{B}$. Clearly $\mathcal{B}'$ forms an optimal complete safe-sweeping process on G' that uses $s(G) - 1$ batteries.

In the latter case, delete u from G and form $\mathcal{B}'$ from $\mathcal{B}$ by placing the same amount of batteries on the remaining vertices and sweeping in the same ordering. Then, $\mathcal{B}'$ is a complete safe-sweeping process on G' that uses $s(G)$ batteries. Note that $\mathcal{B}'$ leaves w with 2 batteries at the end of the process. By Lemma 2.6, $\mathcal{B}'$ is not optimal. Thus, $s(G') < s(G)$. So, in either case $s(G') \leqslant s(G) - 1$.

In the case that $\deg_{G'}(w) \equiv 0 \pmod 3$, we can further reduce the bound to $s(G') \leqslant s(G) - 2$. To see this, suppose that $\deg_{G'}(w) \equiv 0 \pmod 3$. Then, $\deg_G(w) \not\equiv 0 \pmod 3$, and by Observation $(\star)$ we know that w either has at least 1 battery initially or has 1 battery at the end of process $\mathcal{B}$ in G.

Case 1: Suppose there is a battery at w at the end of $\mathcal{B}$. If $\{u, w\}$ is swept from u to w in $\mathcal{B}$, then u initially had 2 batteries. Thus (u, w) is a battery path of w in G. By Lemma 2.9, we can move 2 batteries from u to w and obtain a complete safe-sweeping process for G with the same number of batteries where the edge $\{u, w\}$ is swept from w to u. We may suppose that $\{u, w\}$ is the last edge to be swept in this new process. We can then delete u and remove 2 batteries from w to obtain a complete safe-sweeping process in G' that uses two fewer batteries than $\mathcal{B}$. Otherwise, suppose that $\{u, w\}$ is swept from w to u in $\mathcal{B}$. Let $\mathcal{B}'$ be the complete safe-sweeping process in G' that has the same ordering of edge-sweepings as in $\mathcal{B}$ (apart from the sweeping of $\{u, w\}$). Without loss of generality, we may suppose that $\{u, w\}$ is the last edge that $\mathcal{B}$ sweeps in G; hence, at the end of $\mathcal{B}'$ there are at least 3 batteries at w. If w had 2 or more batteries at the beginning of $\mathcal{B}'$, then delete 2 batteries from w. Otherwise, consider a battery path P for w in $\mathcal{B}'$ along with an ordering of edge-sweepings $\mathcal{E}_{\mathcal{B}',P}$ associated to P in $\mathcal{B}'$ (as defined in the observation preceding Example 2.5). We may suppose that the edges in $\mathcal{E}_{\mathcal{B}',P}$ are swept at the end of process $\mathcal{B}'$. (Note that before any of these edges are swept in $\mathcal{B}'$, there are at least 2 batteries at w.) By Lemma 2.9, we may reverse the directions and the ordering of the edge-sweepings induced by P and form $\mathcal{E}^{-1}_{\mathcal{B}',P}$ accordingly; move 2 batteries from the first vertex of P to w, and then delete 2 batteries from w to obtain a complete safe-sweeping process on G' that uses 2 fewer batteries than $\mathcal{B}$.

Case 2: Now suppose that there is a battery at w at the beginning of $\mathcal{B}$. We may assume that there are no batteries remaining at w at the end of $\mathcal{B}$ (because this situation is handled in Case 1). Since $\deg_{G'}(w) \equiv 0 \pmod 3$, we have $\deg_G(w) \equiv 1 \pmod 3$. Consequently, in G, $in_{\mathcal{B}}(w) \leqslant$

$2 \cdot out_{\mathcal{B}}(w) - 2$ and there are at least 2 batteries at w initially in $\mathcal{B}$. If $\{u, w\}$ is swept from w to u, then delete u and remove 2 batteries from w. It is then easy to see that G' admits a complete safe-sweeping process that uses 2 fewer batteries than $\mathcal{B}$. Otherwise, suppose that $\{u, w\}$ is swept from u to w, then there are 2 batteries at u initially. Delete u along with the 2 batteries it has, and add 1 battery to w to form $\mathcal{B}'$ on G' where all edge-sweepings follow the same ordering as in $\mathcal{B}$ (apart from the sweeping of $\{u, w\}$). Consider an edge e incident with w in G' that is swept away from w in $\mathcal{B}'$. Consider a battery path P in $\mathcal{B}'$ (or a maximal subpath P of a battery path) such that P begins at w and contains e. By the observation preceding Example 2.5, we may suppose that edges in $\mathcal{E}_{\mathcal{B}',P}$ are swept at the beginning of process $\mathcal{B}'$. Remove 3 batteries from w, add 2 batteries to the last vertex of P. Then, applying Lemma 2.9 results in a complete safe-sweeping process that uses 2 fewer batteries than $\mathcal{B}$ does, where the directions and the ordering of the edge-sweepings induced by P are reversed.

For the other direction, let $\mathcal{B}'$ be an optimal complete safe-sweeping process on G' that does not re-sweep edges. Observe that one of three situations occurs: w has 1 battery at the end of the process; w begins the process with at least 1 battery; or w begins and ends with no batteries. In the first situation, we may use the same initial distribution of batteries on G as G', except that we add 1 additional battery to w. The ordering of edge-sweepings may remain the same for G as G', except that the edge $\{u, w\}$ is swept in G at the end (from w to u). Thus, there is a complete safe-sweeping process for G that uses $s(G')+1$ batteries. In the second situation, we may use the same initial distribution of batteries on G as G', except that we move 1 battery from w to u, and add 1 additional battery to u. The ordering of edge-sweepings may remain the same for G as G', except that the edge $\{u, w\}$ is swept in G at the beginning (from u to w). Thus, there is a complete safe-sweeping process for G that uses $s(G') + 1$ batteries. Note that by Observation ($\star$), if $\deg_{G'}(w) \not\equiv 0 \pmod 3$ then the third situation does not occur and the above arguments force $s(G) = s(G') + 1$. However, if $\deg_{G'}(w) \equiv 0 \pmod 3$, then in the third situation we may use the same initial distribution of batteries on G as G', except that we add 2 batteries to u. The ordering of edge-sweepings may remain the same for G as G', except that the edge $\{u, w\}$ is swept in G at the beginning (from u to w). Thus, there is a complete safe-sweeping process for G that uses $s(G') + 2$ batteries; and coupled with an earlier argument, we obtain $s(G) = s(G')+2$ when $\deg_{G'}(w) \equiv 0 \pmod 3$. □

Theorem 3.4. *Let T be a non-empty tree. Then,*

$$s(T) = |E(T)| + \sum_{v \in V(T)} \left\lfloor \frac{deg_T(v) - 1}{3} \right\rfloor + 1.$$

Proof. First, note that the result coincides with Theorem 3.1 if T is a star. Suppose that $s(T') = |E(T')| + \sum_{v \in V(T')} \left\lfloor \frac{deg_{T'}(v)-1}{3} \right\rfloor + 1$ for all trees T' with $k \geqslant 1$ edges. Consider a tree T with $k+1$ edges. Let u be a leaf in T and w be its unique neighbour. Then by Theorem 3.3, $s(T) = |E(T-u)| + \sum_{v \in V(T-u)} \left\lfloor \frac{deg_{T-u}(v)-1}{3} \right\rfloor + 1 + 1$ if $\deg_{T-u}(w) \not\equiv 0 \pmod 3$, and $s(T) = |E(T-u)| + \sum_{v \in V(T-u)} \left\lfloor \frac{deg_{T-u}(v)-1}{3} \right\rfloor + 1 + 2$ if $\deg_{T-u}(w) \equiv 0 \pmod 3$.

Note that $|E(T)| = |E(T-u)| + 1$. Also, if $\deg_{T-u}(w) \not\equiv 0 \pmod 3$, then

$$\sum_{v \in V(T)} \left\lfloor \frac{deg_T(v) - 1}{3} \right\rfloor = \sum_{v \in V(T-u)} \left\lfloor \frac{deg_{T-u}(v) - 1}{3} \right\rfloor;$$

and if $\deg_{T-u}(w) \equiv 0 \pmod 3$, then

$$\sum_{v \in V(T)} \left\lfloor \frac{deg_T(v) - 1}{3} \right\rfloor = \sum_{v \in V(T-u)} \left\lfloor \frac{deg_{T-u}(v) - 1}{3} \right\rfloor + 1.$$

Hence, in either case we obtain

$$s(T) = |E(T)| + \sum_{v \in V(T)} \left\lfloor \frac{deg_T(v) - 1}{3} \right\rfloor + 1. \qquad \square$$

Optimal complete safe-sweeping algorithm for trees

While Theorem 3.4 completely determines the safe-sweep number for trees and along with Theorem 3.3 implicitly describes a procedure for finding an optimal complete safe-sweeping process for trees, it is worthwhile to present an algorithm that does so explicitly. This is accomplished next.

By Lemma 2.11, we may suppose that there are no vertices of degree 2 in T. (Note that contracting degree 2 vertices in a tree does not create loops or multiple edges.) So, suppose that T has at least three edges. Now choose an arbitrary vertex of T to be the root of a rooted tree representation of

T, and give it label 0.1. Arrange the children of the root in such a way as to have a longest root-leaf path lie on the right-most branch of the rooted tree. From left to right, vertices on lower levels will have the same label prefix as their parent vertex but with an additional number at the end that increments. For example, the child vertices of the root will have labels 0.11, 0.12, 0.13, and so on. If a vertex has more than 10 child vertices, change the radix to a larger base in order to accommodate for it. See Figure 4.

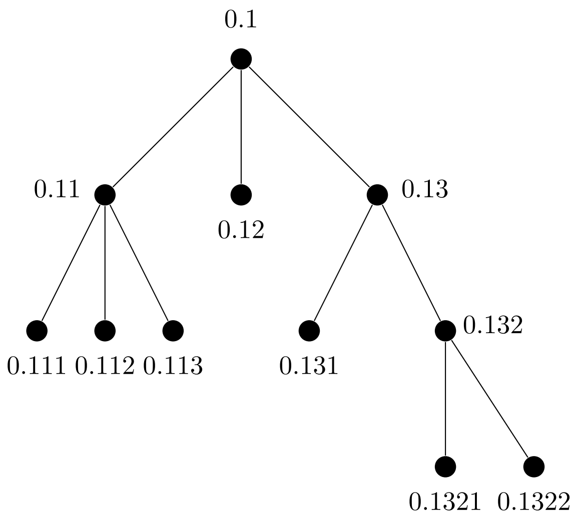

Figure 4: Labelling the vertices of T.

Let $\mathcal{A}$ be the following algorithm. Consider the unswept edge e incident with the leaf with the lowest label, say v, which is adjacent to a vertex u. If u already has 2 batteries and v does not, sweep e from u to v. Otherwise, place enough batteries (0, 1 or 2) on v to be able to sweep e from v to u. When all of the pendant edges incident with u are swept, u is considered a leaf. Repeat the process until all edges have been swept. This safe-sweeping process has a few key properties, which we are listing below as Observation 1, Observation 2 and Observation 3.

Observation 1: Let v be the internal vertex with largest label, and suppose $deg(v) = k + 1$. Then, the last $k + 1$ edges swept by $\mathcal{A}$ induce a star with centre v.

Proof of Observation 1: The result is immediate if T is a star; so, let T be a tree that is not a star.

Suppose there exists a path of length at least 2 from v down to a lower level of the rooted tree. The next vertex on that path after v would be an internal vertex with a larger label than v, contradicting v being the internal vertex with largest label. Therefore, the edges incident with v induce a star where exactly k edges, labelled $e_1, e_2, \ldots, e_k$, are incident with leaves $v_1, v_2, \ldots, v_k$ respectively. Since v is the internal vertex with largest label, one of v_i ($1 \leqslant i \leqslant k$) has the largest label in T. Without loss of generality, assume the labels of $v_1, v_2, \ldots, v_k$ are in ascending order. Label the unique non-leaf vertex adjacent to v as vertex u, and let edge $e = \{u, v\}$.

Let T' be the graph induced by the unswept edges of T at any step in the algorithm $\mathcal{A}$. At each step of $\mathcal{A}$, the lowest labelled leaf in T' is removed from T'. Therefore, T' is never disconnected; it is a tree.

Suppose there exists an edge that is not incident with v such that $\mathcal{A}$ sweeps it after edge e_i for some i, $1 \leqslant i \leqslant k$. Let e' be the first such edge swept by $\mathcal{A}$. The internal vertex incident with e' has a smaller label than v, and the leaf in T' incident with e' has a smaller label than v_i. This contradicts the assumption that e' would be swept after e_i for some i, $1 \leqslant i \leqslant k$. Hence, it must be the case that the last k edges swept by $\mathcal{A}$ are the edges incident with v and a leaf.

Now consider the step where e is swept. Either u or v must be a leaf in T' at this point. If v is a leaf, then $\mathcal{A}$ must have swept $e_1, \ldots, e_k$ before sweeping e, contradicting our result that the edges incident to a leaf and v are swept last. So, u is a leaf in T' (and v is not). Then, e is the last edge $\mathcal{A}$ sweeps before sweeping $e_1, \ldots, e_k$, because otherwise sweeping e would disconnect T'.

We conclude that all $k + 1$ edges incident with v are collectively the last edges $\mathcal{A}$ sweeps and they form a star.

Observation 2: Let v be the internal vertex in T with the largest label. Then, the last edge swept by $\mathcal{A}$ is $e = \{v, x\}$ where x is the leaf with the largest label in T, and it is swept towards x.

Proof of Observation 2: By Observation 1, the star induced by the edges incident with v are the last edges swept by $\mathcal{A}$. Consequently, $e = \{v, x\}$ is the last edge swept. When all the other edges are swept, v becomes a leaf in T'. Since x does not receive batteries from prior edge-sweepings in $\mathcal{A}$ and v has smaller label than x, $\mathcal{A}$ will sweep e from v towards x.

Observation 3: Let v be the internal vertex in T with the largest label, and u be the vertex with the smallest label adjacent to v. Then, $\mathcal{A}$ sweeps $e = \{u, v\}$ from u towards v.

Proof of Observation 3: From Observation 1, $e = \{u, v\}$ is the first edge incident with v that is swept. Since v is not a leaf at the step when e is swept and no other edges incident with v have been swept, v currently does not have 2 or more batteries. Therefore, $\mathcal{A}$ will sweep e from u to v.

We shall now proceed by induction on the number of edges of T to show that $\mathcal{A}$ uses only $|E(T)| + \sum_{v\in V(T)} \left\lfloor \frac{deg(v)-1}{3} \right\rfloor + 1$ batteries. We will denote by $s_{\mathcal{A}}(T)$ the number of batteries used in the algorithm $\mathcal{A}$ until all edges of T are swept.

Suppose that $\mathcal{A}$ produces a complete safe-sweeping process using exactly $|E(T)| + \sum_{v\in V(T)} \left\lfloor \frac{deg(v)-1}{3} \right\rfloor + 1$ batteries for any tree T with $m \geqslant 3$ edges. Consider a tree T with $m+1$ edges. Let x be the leaf with the largest label and v be the internal vertex adjacent to x. Suppose v has degree $k+1$. By Observation 2, $\{v, x\}$ will be the last edge swept by $\mathcal{A}$.

$T - x$ is a tree with m edges. Suppose w is the leaf with the largest label in $T - x$. By the induction hypothesis,

$$\begin{aligned} s_{\mathcal{A}}(T-x) &= |E(T-x)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1 \\ &= |E(T)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor. \end{aligned}$$

By Observation 3, the first edge incident with v in T to be swept will be swept towards v. In what follows, we may consider that edge to have been swept and hence bringing one battery to v.

By Observation 1, the last k edges swept in $T - x$ form a star at v. It will suffice to consider the congruence classes of k modulo 3 to see by how much $s_{\mathcal{A}}(T)$ differs from $s_{\mathcal{A}}(T - x)$. Note that for every 2 edges that $\mathcal{A}$ sweeps towards v, the next pendant edge incident with v is swept from v.

Case 1: Suppose $k \equiv 0 \pmod 3$. There will be zero batteries left at v after the sweeping of all pendant edges incident with v in $T - x$. Hence in

T, $\mathcal{A}$ will place 2 batteries at v to sweep the edge incident with x. Thus,

$$\begin{aligned} s_{\mathcal{A}}(T) &= s_{\mathcal{A}}(T-x)+2 \\ &= |E(T-x)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1 + 2 \\ &= |E(T)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 2. \end{aligned}$$

Observe that $\left\lfloor \frac{k}{3} \right\rfloor - 1 = \left\lfloor \frac{k-1}{3} \right\rfloor$, when $k \equiv 0 \pmod 3$; so,

$$\sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor = \sum_{v\in V(T)} \left\lfloor \frac{deg_{T}(v)-1}{3} \right\rfloor - 1,$$

giving us $s_{\mathcal{A}}(T) = |E(T)| + \sum_{v\in V(T)} \left\lfloor \frac{deg_T(v)-1}{3} \right\rfloor + 1$ as desired.

Case 2: Suppose $k \equiv 1 \pmod 3$. In $T-x$, $\mathcal{A}$ places 2 batteries at v in order to sweep the edge incident with w (from v to w). In T, since v is also adjacent with x, those batteries are instead placed at w and the edge is swept towards v. This means $\mathcal{A}$ will only have to place 1 battery at v to sweep the last edge from v to x. Thus,

$$\begin{aligned} s_{\mathcal{A}}(T) &= s_{\mathcal{A}}(T-x)+1 \\ &= |E(T-x)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1 + 1 \\ &= |E(T)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1. \end{aligned}$$

Observe that $\left\lfloor \frac{k}{3} \right\rfloor = \left\lfloor \frac{k-1}{3} \right\rfloor$, when $k \equiv 1 \pmod 3$; so,

$$\sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor = \sum_{v\in V(T)} \left\lfloor \frac{deg_{T}(v)-1}{3} \right\rfloor,$$

giving us $s_{\mathcal{A}}(T) = |E(T)| + \sum_{v\in V(T)} \left\lfloor \frac{deg_T(v)-1}{3} \right\rfloor + 1$ as desired.

Case 3: Suppose $k \equiv 2 \pmod 3$. In $T-x$, $\mathcal{A}$ places 1 battery at v to sweep from v towards w since at that step v has already received a battery from an earlier edge-sweeping. In T, since v is also adjacent to x, $\mathcal{A}$ places 2 batteries at w and sweeps from w towards v. The edge $\{v, x\}$ is then swept at no additional cost from v to x since v already received 2 batteries from earlier edge-sweepings. Thus,

$$\begin{aligned} s_{\mathcal{A}}(T) &= s_{\mathcal{A}}(T-x)+1 \\ &= |E(T-x)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1 + 1 \\ &= |E(T)| + \sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor + 1. \end{aligned}$$

Observe that $\left\lfloor \frac{k}{3} \right\rfloor = \left\lfloor \frac{k-1}{3} \right\rfloor$ when $k \equiv 2 \pmod 3$; so,

$$\sum_{v\in V(T-x)} \left\lfloor \frac{deg_{T-x}(v)-1}{3} \right\rfloor = \sum_{v\in V(T)} \left\lfloor \frac{deg_{T}(v)-1}{3} \right\rfloor,$$

giving us $s_{\mathcal{A}}(T) = |E(T)| + \sum_{v\in V(T)} \left\lfloor \frac{deg_T(v)-1}{3} \right\rfloor + 1$ as desired.

We finish with a simple result where we utilize edge-decompositions to obtain an upper bound on $s(G)$ for any simple graph G. First we need the following well-known theorem, a proof of which can be found for example on page 29 of [9].

Theorem 3.5. *A connected graph with $2k$ odd vertices admits an edge-decomposition into k edge-disjoint trails.*

Theorem 3.6. *Let G be a graph with $2k$ ($k \geqslant 1$) vertices of odd degree. Then, $s(G) \leqslant |E(G)| + k$.*

Proof. Let G be a graph with $2k$ vertices of odd degree. By Theorem 3.5, G can be decomposed into k edge-disjoint trails $W_1, \ldots, W_k$. For each i ($1 \leqslant i \leqslant k$), place 2 batteries at one of the two end-vertices of W_i and 1 battery at each interior vertex of W_i; and sweep edges along W_i starting at the vertex that received 2 batteries. Doing so produces a complete safe-sweeping process of G with $|E(G)| + k$ batteries. □

Note that this bound is clearly not tight. For example, it implies that $s(K_{1,5}) \leqslant 8$; however, $s(K_{1,5}) = 7$ by Theorem 3.1.

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