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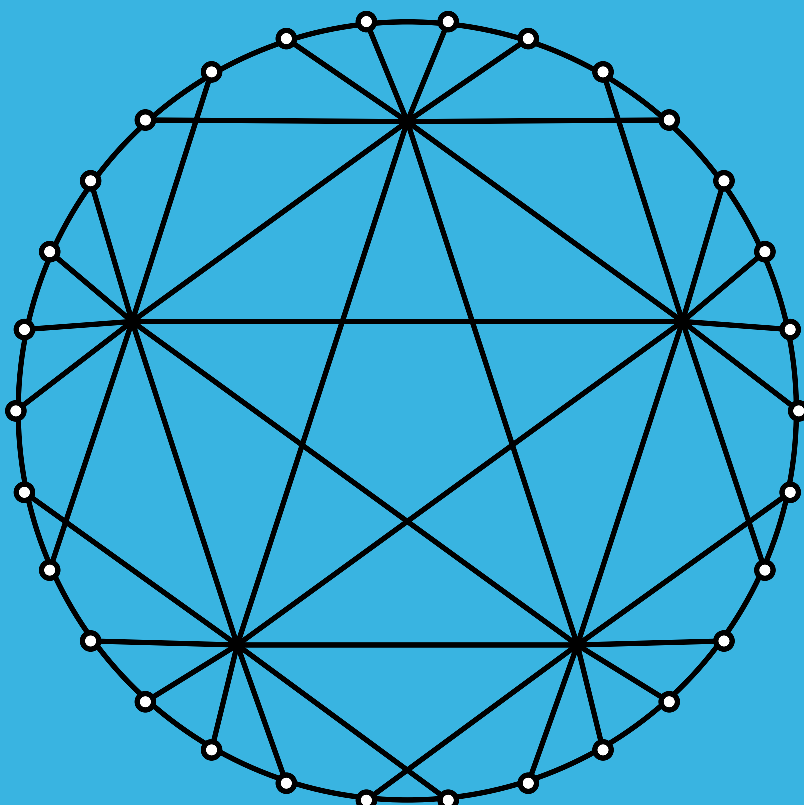

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Odd harmonious labeling of the converse skew product of graphs

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Abstract. In this paper, we utilize the ideas of odd harmonious labeling and the converse skew product to obtain more odd harmonious graphs. Also, we investigate its α-labeling. Finally, we define with proof a necessary condition for preserving odd graceful.

1 Introduction and preliminaries

A graph that has order n and size m is called a (n, m)-graph. Let $G(n, m)$ be a graph. An injective vertex function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is said to be graceful [13] if $f^*(uv) = |f(u) - f(v)|$ from $E(G)$ to $\{1, 2, 3, \ldots, m\}$ is injective. A graph that admits a graceful labeling is called graceful graph. A graceful graph G is said to be α-valuable if it has a graceful labeling f such that for some positive integer λ either $f(u) \leq \lambda$ and $f(v) > \lambda$ or $f(u) > \lambda$ and $f(v) \leq \lambda$ for every edge $uv \in E(G)$. λ is said to be the characteristic of f. Gnanajothi [6] defined a graph $G(n, m)$ to be odd graceful if there is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2m - 1\}$ such that, if the label of each edge uv is defined as $f(uv) = |f(u) - f(v)|$, the set of all edge labels is equal to $\{1, 3, 5, \ldots, 2m - 1\}$. A function f is said to be an odd harmonious [10] labeling of a graph G with m edges if f is an injection from the vertices of G to the integers from 0 to $2m - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2m-1$ is a bijection. If $G(n, m)$ is an odd harmonious graph with odd labeling f such that $f(u) \leq m$, for all $u \in V(G)$, then G is called strongly odd harmonious and f is called a strongly odd harmonious labeling of G. In [14], Seoud and Hafez proved that $\alpha-$valuable graphs are odd harmonious and strongly odd harmonious graphs are $\alpha-$valuable graphs. Let $G(n, m)$ be a graph that admits a strongly odd harmonious

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labeling f. Then the mapping $g : V(G) \to \{0, 1, 2, \ldots, m\}$ defined by

$$g(u) = \begin{cases} \frac{f(u)}{2} & f(u) \text{ is even} \\ m - \frac{f(u)-1}{2} & f(u) \text{ is odd} \end{cases} \quad (1)$$

is an α-labeling for G, see [14]. Conversely, if G is a bipartite graph with bipartition (A, B) and f is an α-labeling of G such that $f(u) \leq \lambda$ for all $u \in A$, then the mapping

$$g(u) = \begin{cases} 2(\lambda - f(u)) & u \in A \\ 2(f(u) - \lambda) - 1 & u \in B \end{cases} \quad (2)$$

is an odd harmonious labeling of G, see [3, 14].

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The converse skew product $G \bigtriangledown H$ [15] has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[y_1 = y_2$ and $x_1x_2 \in E(G)]$ or $[x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)]$. The symmetric difference $G \oplus H$ has $V(G) \times V(H)$ as its vertex set and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $x_1x_2 \in E(G)$ or $y_1y_2 \in E(H)$ but not both. The composition $G[H]$ has its vertex set $V(G) \times V(H)$ and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[x_1x_2 \in E(G)]$ or $[x_1 = x_2$ and $y_1y_2 \in E(H)]$. According the definition of the converse skew product, $G(n_1, m_1) \bigtriangledown H(n_2, m_2)$ has $m_1(2m_2 + n_2)$ edges. The $m-$splitting graph [1], $Spl_m(G), m \geq 1$, of a graph G is the graph obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, \ldots, v_m$, such that $v_i, 1 \leq i \leq m$, is adjacent to every vertex that is adjacent to v in G. The $m-$shadow graph [1], $D_m(G), m \geq 1$, of a graph G is obtained by taking m copied of G say $G_1, G_2, \ldots, G_m$, and then join each vertex u in G_i to the neighbours of the corresponding vertex v in $G_j, 1 \leq j \leq m$.

Vaidya [17] proved that $D_2(G)$ is odd harmonious whenever G is P_n and $K_{1,n}$. Jeyanthi and Philo [7] proved that $D_2(G)$ is odd harmonious whenever G is $K_{2,n}$ and C_{4m}. Abdel-Aal and Seoud [2] proved that the graph $D_m(G)$ is odd harmonious whenever G is $P_n, K_{m,n}, D_2(P_n)$. Jeyanthi and Philo [8] proved that the graph $D_m(G)$ is odd harmonious whenever G is $P_n, K_{m,n}, D_2(P_n), C_{4m}$. Sarah Minion and Christian Barrientos [11] proved that $G \oplus \overline{K_2} = G[\overline{K}_2]$ has an α-labeling whenever G has an α-labeling. Abdel-Aal [1] proved that the graph $Spl_m(G)$ of a graph G is odd graceful whenever G is $P_n, K_{m,n}, D_2(P_n), C_{4m}$. Sathiamoorthy, Natarajan, Ayyaswamy, and Janakiraman [4] proved that the splitting graph of a caterpillar graph is graceful. For a dynamic survey of graph labeling, we refer to [5].

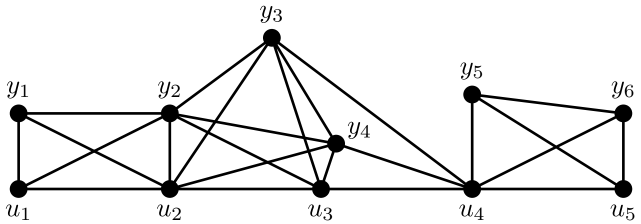


Figure 1: $K(5)$: $V(K_{r_1}) = \{y_1, y_2\}$, $V(K_{r_2}) = \{y_2, y_3, y_4\}$, $V(K_{r_3}) = \{y_3, y_4\}$, and $V(K_{r_4}) = \{y_6, y_7\}$

Denote the vertices of $P_n, n \geq 2$, by $u_1, u_2, \ldots, u_n$ such that

$$u_i u_{i+1}, i = 1, 2, \ldots, n-1,$$

is an edge. Throughout the paper, let $\{K_j(n), n \geq 2\}$ be the class of graphs obtained from P_n by joining the end vertices of the edge $u_i u_{i+1}$ to every vertex in the complete graph $K_{r_i}, i = 1, 2, \ldots, n-1$, of order r_i, such that $|V(K_{r_i}) \cap V(K_{r_{i+1}})| = t$, where $t = 0, 1, 2, \ldots, Min\{r_i, r_{i+1}\}$ and $V(K_{r_i}) \cap V(K_{r_j}) = \varnothing, |i - j| \geq 2$, see Figure 1. For simplicity, we denote $V(k_{r_i})$ by V_i and refer to elements in $\{K_j(n), n \geq 2\}$ by $K(n)$. Note that $|V(K(n))| = \sum_{i=1}^{i=n-1} r_i + n - \sum_{i=1}^{i=n-2} |V_i \cap V_{i+1}|$ and $|E(K(n))| = \sum_{i=1}^{i=n-1} \binom{r_i+2}{2} - \sum_{i=1}^{i=n-2} \binom{|V_i \cap V_{i+1}|+1}{2}$.

We note that $D_n(G) = G[\overline{K_n}] = G \bigtriangledown K_n = G \bigtriangledown K(2)$, where $K(2)$ is obtained form P_2 by joining u_1 and u_2 to every vertex in the complete graph K_{n-2}. In the next section, we prove that the graph $G \bigtriangledown K(n), n \geq 2$ is odd harmonious whenever G is odd harmonious. This result generalizes the results in [2, 7, 8, 11, 17].

2 Main Results

Theorem 2.1. *If G is odd harmonious, then the graph $G \bigtriangledown K(n)$ is odd harmonious.*

Proof. Let $G(p, m)$ be an odd harmonious graph with odd harmonious labeling f. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and V_e, V_o be the partite sets of $V(G)$ such that $V_e = \{u_j \in V(G) : f(u_j) \text{ is even}\}$ and $V_o = \{u_j \in V(G) : f(u_j) \text{ is odd}\}$. Denote the vertices of $K(n)$ by $y_0, y_1, \ldots, y_{a-1}$, where a is the number of vertices of $K(n)$, in such a manner that the vertices of P_n take the following labels $y_0, y_{r_1+1}, y_{r_1+r_2+\ldots+r_i+i-(|V_1\cap V_2|+|V_2\cap V_3|+\ldots+|V_i\cap V_{i-1}|)}, i = 2,3,\ldots,n-1$ and the vertices in $V_i \cap V_{i+1}$ are assigned labels with higher subscripts than the labels assigned to the vertices in $V_i \backslash V_i \cap V_{i+1}$, see Figure 2. Recall that $G \bigtriangledown K(n)$ can be constructed by taking a copies of G say $G^0, G^1, G^2, \ldots, G^{a-1}$ and join each vertex u in copy G^i to the neighbors of the corresponding vertex in copy G^j whenever y_i and y_j are adjacent in $K(n)$. In $G \bigtriangledown K(n)$, denote the partite sets of copy G^i by V_e^i and V_o^i. For a vertex $u_j \in V(G)$, we denote the corresponding vertex in copy G^i by u_{ij}. Let g be the vertex labeling on $V(G \bigtriangledown K(n))$ which is defined by:

$$g(u_{ij}) = \begin{cases} f(u_j) + 2im & u_{ij} \in V_e^i, \text{ for } i = 0,1,\cdots,a-1 \\ f(u_j) + 2(r_1+2)im & u_{ij} \in V_o^i \text{ and } i \in N_1 \\ g(u_{i-1j}) + 2(r_t + r_{t+1} \\ \qquad +3 - |V_t \cap V_{t+1}|)m & u_{ij} \in V_o^i \text{ and } i \in N_2 \\ g(u_{i-1j}) + 2(r_{t+1}+2)m & u_{ij} \in V_o^i \text{ and } i \in N_3, \end{cases} \tag{3}$$

where $N_1 = \{0,1,\ldots,r_1+1-|V_1 \cap V_2|\}$, N_2 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}|) \right., \\ \left. \sum_{j=1}^{j=t} r_j + t - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|) \right],$$

for $t = 1,2,\ldots,n-2$ and $|V_t \cap V_{t+1}| \neq 0$, and N_3 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|) \right., \\ \left. \sum_{j=1}^{j=t+1} r_j + t + 1 - (\sum_{j=1}^{j=t+1} |V_j \cap V_{j+1}|) \right],$$

for $t = 1,2,\ldots,n-2$.

g is injective and

$$g(u_{ij}) \leq 2|E\left(G \bigtriangledown K(n)\right)| - 1 = 2m\left(2|E(K(n))| + |V(K(n))|\right) - 1.$$

We will show that the induced edge labeling g^* defined from $E(G \bigtriangledown K(n))$ to $2m(2|V(K(n))| + |E(K(n))|) - 1$ by $g^*(u_{i_1j_1}u_{i_2j_2}) = g(u_{i_1j_1}) + g(u_{i_2j_2})$ is injective. According to the construction of $G \bigtriangledown K(n)$, we have to show that:

- Case 1: edges in $G^i, i = 0, 1, 2, \ldots, a-1$, have distinct labels.
- Case 2: edges in G^i and G^j, have distinct labels for $i \neq j = 0, 1, 2, \ldots, a-1$, have distinct labels.
- Case 3: edges joining vertices in G^i to the neighbors of the corresponding vertex in copy G^j whenever y_i and y_j are adjacent in $K(n)$, have distinct labels.
- Case 4: edges in case 3 and case 1 have distinct labels.

Using 3, vertices in $V_o^i, i \in N_2, t = s, s = 1, 2, 3, .., n-2$ have labels of the form:

$$\begin{aligned}
g(u_{ij}) = {} & f(u_j) + 2m(r_1+2)\left(r_1+1-|V_1 \cap V_2|\right) \\
& + \sum_{k=1}^{k=s-1} 2m\left(r_k + r_{k+1} + 3 - |V_k \cap V_{k+1}|\right)\left(|V_k \cap V_{k+1}|\right) + \\
& + \sum_{k=1}^{k=s-1} 2m\left(r_{k+1}+2\right)\left(r_{k+1}+1-\left(|V_k \cap V_{k+1}| + |V_{k+1} \cap V_{k+2}|\right)\right) \\
& + 2m\left(i+1-\left(\sum_{k=1}^{k=s} r_k + s + 1 - (\sum_{k=1}^{k=s} |V_k \cap V_{k+1}|)\right)\right) \\
& \qquad\qquad (r_s + r_{s+1} + 3 - (V_s \cap V_{s+1}))
\end{aligned} \tag{4}$$

and vertices in $V_o^i, i \in N_3, t = s, s = 1, 2, 3, \ldots, n-2$ have labels of the form:

$$\begin{aligned}
g(u_{ij}) = {} & f(u_j) + 2m(r_1+2)\left(r_1+1-|V_1 \cap V_2|\right) \\
& + \sum_{k=1}^{k=s} 2m\left(r_k + r_{k+1} + 3 - |V_k \cap V_{k+1}|\right)\left(|V_k \cap V_{k+1}|\right) \\
& + \sum_{k=1}^{k=s-1} 2m\left(r_{k+1}+2\right)\left(r_{k+1}+1-\left(|V_k \cap V_{k+1}| + |V_{k+1} \cap V_{k+2}|\right)\right) \\
& + 2m\left(i+1-\left(\sum_{j=1}^{j=s} r_j + s + 1 - (\sum_{j=1}^{j=s} |V_j \cap V_{j+1}| - |V_s \cap V_{s+1}|)\right)\right) \\
& \qquad\qquad (r_{s+1}+2)
\end{aligned} \tag{5}$$

Cases 1 and 2 are obvious.

Case 3: assume that two edges $u_{i_1j_1}u_{i_2j_2}$ and $u_{i_3j_3}u_{i_4j_4}$ have the same label. From the construction of $G \bigtriangledown K(n)$, we note that:

- If $i_1 \in N_1$, then i_2 must be in $N_1, N_2, t = 1, 2$, or $N_3, t = 1$.
- If $i_1 \in N_2, t = 1$, then i_2 must be in $N_1, N_2, t = 1, 2$, or $N_3, t = 1$.
- If $i_1 \in N_3, t = 1$, then i_2 must be in $N_1, N_2, t = 1, 2, 3$, or $N_3, t = 1, 2$.
- If $i_1 \in N_2, t = i, i = 2, 3, \ldots, n-2$, then i_2 must be in $N_2, t = i, i-1, i+1$ or in $N_3, t = i, i-1, i-2$.
- If $i_1 \in N_3, t = i, i = 2, 3, \ldots, n-2$, then i_2 must be in $N_2, t = i, i+1, i+2$ or in $N_3, t = i, i-1, i+1$.

Both i_3 and i_4 have the same possible choices as i_1 and i_2. Without loss of generality, assume that $u_{i_1j_1} \in V_e^{i_1}$ and $u_{i_3j_3} \in V_e^{i_3}$. Although there are many subcases, they have the same method to prove them. So, we will prove some subcases and the other subcases follow immediately as we will clarify. Assume that $i_1, i_2, i_3, i_4 \in N_1$. We know that $i_1 \neq i_2 \neq i_3 \neq i_4$ and $u_{j_1}u_{j_2}$ and $u_{j_3}u_{j_4}$ are edges in G. Now we have

$$f(u_{j_1})+2i_1m+f(u_{j_2})+2(r_1+2)i_2m = f(u_{j_3})+2i_3m+f(u_{j_4})+2(r_1+2)i_4m$$

Therefore,

$$f(u_{j_1}) + f(u_{j_2}) - (f(u_{j_3}) + f(u_{j_4})) = 2m(i_3 - i_1) + 2(r_1 + 2)m(i_4 - i_3) \quad (6)$$

Since f is an odd harmonious labeling, then

$$|f(u_{j_1}) + f(u_{j_2}) - (f(u_{j_3}) + f(u_{j_4}))| \leq 2m \quad (7)$$

We claim that $2m(i_3 - i_1) + 2(r_1 + 2)m(i_4 - i_3)$ does not belong to the $[-2m, 2m]$. Assume on contradiction that $-2m \leq 2m(i_3 - i_1) + 2(r_1 + 2)m(i_4 - i_3) \leq 2m$. Then $-1 \leq (i_3 - i_1) + (r_1 + 2)(i_4 - i_3) \leq 1$, which imply that $(i_3 - i_1) \leq (r_1 + 2)(i_4 - i_3) \leq (i_1 - i_3)$. In other words, $|(r_1 + 2)(i_4 - i_3)| \leq |(i_1 - i_3)|$. Since $|(i_1 - i_3)| \leq r_1 + 1 - |V_1 \cap V_2|$, then $|(r_1 + 2)(i_4 - i_3)| \leq r_1 + 1 - |V_1 \cap V_2|$, a contradiction.

For subcases when i_1, i_2, i_3, or i_4 take small values or all in $N_2, t = i$ or $N_3, t = i$, the proof is similar to the subcase when they all in N_1, since the right-hand side of Equation 6 would not change much more. So, assume that $i_1 \in N_2, t = i$ and $i_3 \in N_2, t = s$ and $s \neq t$. Then i_2 should be in $N_2, t = i, i-1, i+1$ or in $N_3, t = i, i-1, i-2$. i_4 should be in $N_2, t =$

$s, s-1, s+1$ or in $N_3, t = s, s-1, s-2$. Assume that $i_2 \in N_2, t = i$ and $i_4 \in N_3, t = s$ and $i < s$. The right-hand side of Equation 6 would be :

$$\begin{aligned}
&\sum_{k=i}^{k=s} 2m\left(r_k + r_{k+1} + 3 - |V_k \cap V_{k+1}|\right)\left(|V_k \cap V_{k+1}|\right) + \\
&\sum_{k=i}^{k=s-1} 2m\left(r_{k+1} + 2\right)\left(r_{k+1} + 1 - \left(|V_k \cap V_{k+1}| + |V_{k+1} \cap V_{k+2}|\right)\right) + \\
&2m\left(i_4 + 1 - \left(\sum_{j=1}^{j=s} r_j + s + 1 - (\sum_{j=1}^{j=s} |V_j \cap V_{j+1}| - |V_s \cap V_{s+1}|)\right)\right) \\
&(r_{s+1} + 2) - 2m\left(i_2 + 1 - \left(\sum_{k=1}^{k=i} r_k + i + 1 - (\sum_{k=1}^{k=i} |V_k \cap V_{k+1}|)\right)\right) \\
&\qquad\qquad (r_i + r_{i+1} + 3 - (|V_i \cap V_{i+1}|))
\end{aligned} \tag{8}$$

Note that

$$\left(i_2 + 1 - \left(\sum_{k=1}^{k=i} r_k + i + 1 - (\sum_{k=1}^{k=i} |V_k \cap V_{k+1}|)\right)\right) \leq |V_i \cap V_{i+1}|$$

and

$$\begin{aligned}
&\left(i_4 + 1 - \left(\sum_{j=1}^{j=s} r_j + s + 1 - (\sum_{j=1}^{j=s} |V_j \cap V_{j+1}| - |V_s \cap V_{s+1}|)\right)\right) \\
&\qquad \leq r_{s+1} + 1 - \left(|V_s \cap V_{s+1}| + |V_{s+1} \cap V_{s+2}|\right).
\end{aligned}$$

Hence it is clear that the absolute value of 8 is much more than $2m$ and 8 cannot be in $[-2m, 2m]$. For any other choices for i_1, i_2, i_3, and i_4, by rearranging terms in 6, we can get a right-hand side that is similar to the quantity 8 which could not be in the interval $[-2m, 2m]$.

Case 4. Assume that two edges $u_{i_1j_1}u_{i_1j_2} \in E\left(G^{i_1}\right)$ and $u_{i_2j_3}u_{i_3j_4}$, joining $u_{i_2j_3} \in G^{i_2}$ and $u_{i_3j_4} \in G^{i_3}$, have the same label. Assume, without loss of generality, that $u_{i_1j_1} \in V_e^{i_1}$ and $u_{i_2j_3} \in V_e^{i_2}$. The choices of i_2 and i_3 are the same as the choices of i_1 and i_2 in Case 3, therefore we will prove some subcases and the other subcases follow immediately. If $i_1 \in N_1$, $i_2 \in N_2, t = i$, and $i_3 \in N_3, t = i$, then we would have

$$f(u_{j_1}) + 2i_1m + f(u_{j_2}) + 2(r_1 + 2)i_1m = f(u_{j_3}) + 2i_2m + g(u_{i_3j_4})\cdot$$

Therefore using 3 and 5,

$$f(u_{j_1}) + f(u_{j_2}) - (f(u_{j_3}) + f(u_{j_3})) = A, \tag{9}$$

where

$$\begin{aligned}
A =& 2m(i_2 - i_1) + 2m(r_1 + 2)(r_1 - i_1 + 1 - |V_1 \cap V_2|) + \\
& \sum_{k=1}^{k=i} 2m(r_k + r_{k+1} + 3 - |V_k \cap V_{k+1}|)(|V_k \cap V_{k+1}|) + \\
& \sum_{k=1}^{k=i-1} 2m(r_{k+1} + 2)(r_{k+1} + 1 - (|V_k \cap V_{k+1}| + |V_{k+1} \cap V_{k+2}|)) + \\
& 2m\left(i_3 + 1 - \left(\sum_{j=1}^{j=i} r_j + i + 1 - (\sum_{j=1}^{j=i} |V_j \cap V_{j+1}| - |V_i \cap V_{i+1}|)\right)\right)(r_{i+1} + 2),
\end{aligned} \tag{10}$$

which could not be in $[-2m, 2m]$. For any other choice of i_2 and i_3 with $i_1 \in N_1$, we get the same result. If $i_1 \in N_2, t = s, i_2 \in N_2, t = i$, and $i_3 \in N_3, t = i$ and $s < i$, then (**??**) would be

$$\begin{aligned}
& \sum_{k=s+1}^{k=i} 2m(r_k + r_{k+1} + 3 - |V_k \cap V_{k+1}|)(|V_k \cap V_{k+1}|) + \\
& \sum_{k=s}^{k=i-1} 2m(r_{k+1} + 2)(r_{k+1} + 1 - (|V_k \cap V_{k+1}| + |V_{k+1} \cap V_{k+2}|)) + \\
& 2m\left(i_4 + 1 - \left(\sum_{j=1}^{j=s} r_j + s + 1 - (\sum_{j=1}^{j=s} |V_j \cap V_{j+1}| - |V_s \cap V_{s+1}|)\right)\right) \\
& (r_{s+1} + 2) - 2m\left(i_2 + 1 - \left(\sum_{k=1}^{k=s} r_k + s + 1 - (\sum_{k=1}^{k=s} |V_k \cap V_{k+1}|)\right)\right) \\
& (r_s + r_{s+1} + 3 - (|V_s \cap V_{s+1}|)),
\end{aligned} \tag{11}$$

which again could not be in $[-2m, 2m]$. For any other choices for i_1, i_2, and i_3, by rearranging terms in 9, we can get a right-hand side that is similar to the quantity 11.

To complete the proof, it is sufficient to prove that the label assigned by g^* to an edge is at most $2|E\left(G \bigtriangledown K(n)\right)|-1 = 2m\left(2|E(K(n))| + |V(K(n))|\right)-1$, indeed edge in G^{a-1} that is corresponding to the edge labeled $2m-1$ in G is assigned the label, using 3 and 5:

$$\begin{aligned}
&2m-1+2m(a-1)+2m(r_1+2)\left(r_1+1-|V_1\cap V_2|\right)+\\
&\sum_{k=1}^{k=n-2} 2m\left(r_k+r_{k+1}+3-|V_k\cap V_{k+1}|\right)\left(|V_k\cap V_{k+1}|\right)+\\
&\sum_{k=1}^{k=n-2} 2m\left(r_{k+1}+2\right)\left(r_{k+1}+1-\left(|V_k\cap V_{k+1}|+|V_{k+1}\cap V_{k+2}|\right)\right)\cdot
\end{aligned}$$

Note that $a = |V(K(n))|$ and

$$|E(K(n))| = \sum_{i=1}^{i=n-1}\binom{r_i+2}{2} - \sum_{i=1}^{i=n-2}\binom{|V_i\cap V_{i+1}|+1}{2}.$$

It is not difficult to prove that:

$$\begin{aligned}
2|E(K(n))| = {}&(r_1+2)\left(r_1+1-|V_1\cap V_2|\right)+\\
&\sum_{k=1}^{k=n-2}\left(r_k+r_{k+1}+3-|V_k\cap V_{k+1}|\right)\left(|V_k\cap V_{k+1}|\right)+\\
&\sum_{k=1}^{k=n-2}\left(r_{k+1}+2\right)\left(r_{k+1}+1-\left(|V_k\cap V_{k+1}|+|V_{k+1}\cap V_{k+2}|\right)\right)
\end{aligned}\tag{12}$$

Therefore, g^* is bijective and g is an odd harmonious labeling of $G \bigtriangledown K(n)$. □

To understand the distribution of edge labels given the proof of Theorem 2.1, denote the labels assigned to the edges in copy $G^i, i = 0,1,2,\ldots,a-1$ by I_i. Labels in $I_0 = \{1,3,5,\ldots,2m-1\}$ are assigned to the edges in G^0. The labels in the interval that is between the intervals I_i and I_{i+1}, are assigned to the edges joining the vertices in V_o^i to the vertices in V_e^{i+1}, V_e^{i+2}, ..., and V_e^s , for all $i \le s$ such that y_s and y_i are adjacent in $K(n)$ and the edges joining the vertices in V_o^{i+1} to the vertices in $V_e^i, V_e^{i-1},\ldots,$ and V_e^s, for all $s \le i+1$ such that y_s and y_i are adjacent in $K(n)$, see the next Example 2.1.1.

Example 2.1.1. *Let G be the path on 4 vertices, P_4, with the odd harmonious labeling $[5,0,1,2]$ for $[u_1,u_2,u_3,u_4]$. Figure 3 shows the odd harmonious labeling of $P_4 \bigtriangledown K(5)$, where $K(5)$ is the graph in Figure 2. Edges* *in G^0 are assigned the labels in $I_0 = \{1, 3, 5\}$, then the edges joining vertices in V_o^0 to vertices in V_e^1, V_e^2, and V_e^3 and the edges joining vertices in V_o^1 to vertices in V_e^0 are assigned the next set of edge labels, that is $\{7, 9, 11, \ldots, 29\}$. Then edges in G^1 are assigned the next set of edge labels, that is $I_1 = \{31, 33, 35\}$, then the edges joining vertices in V_o^1 to vertices in V_e^2 and V_e^3 and the edges joining vertices in V_o^2 to vertices in V_e^0 and V_e^1 are assigned the next set of edge labels, that is $\{37, 39, \ldots, 59\}$. Then edges in G^2 are assigned the next set of edge labels, that is $I_2 = \{61, 63, 65\}$, then the edges joining vertices in V_o^2 to vertices in V_e^3, V_e^4, V_e^5, and V_e^6 and the edges joining vertices in V_o^3 to vertices in V_e^0, V_e^1, and V_e^2 are assigned the next set of edge labels, that is $\{67, 69, \ldots, 107\}$, and so on.*

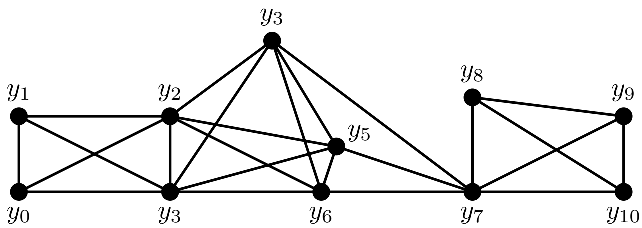


Figure 2: Labeling of $K(5)$ as it is described in the proof of Theorem 2.1

Corollary 2.1.1. *If G is odd harmonious, then $G[\overline{K}_r]$ is odd harmonious, for all $r \geq 2$.*

Proof. The graph $G[\overline{K}_r]$ has r^2m edges and rn vertices. Since $G[\overline{K}_r] = G \bigtriangledown K_r = G \bigtriangledown K(2)$, where $K(2)$ is obtained form P_2 by joining u_1 and u_2 to every vertex in the complete graph K_{r-2}, the graph $G[\overline{K}_r]$ is odd harmonious. According to Theorem 2.1, the function $g : V(G[\overline{K}_r]) \to \{0, 1, 2, \ldots, 2r^2m - 1\}$ defined by:

$$g(u_{ij}) = \begin{cases} f(u_j) + 2im & u_{ij} \in V_e^i \\ f(u_j) + 2rim & u_{ij} \in V_o^i \end{cases} \tag{13}$$

is odd harmonious. □

Theorem 2.2. *If G has an α-labeling, then the graph $G \bigtriangledown K(n)$ has an α-labeling.*

Proof. Let $G(p, m)$ be a bipartite graph with bipartition (A, B) and f be an α-labeling of G with characteristic λ. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ such

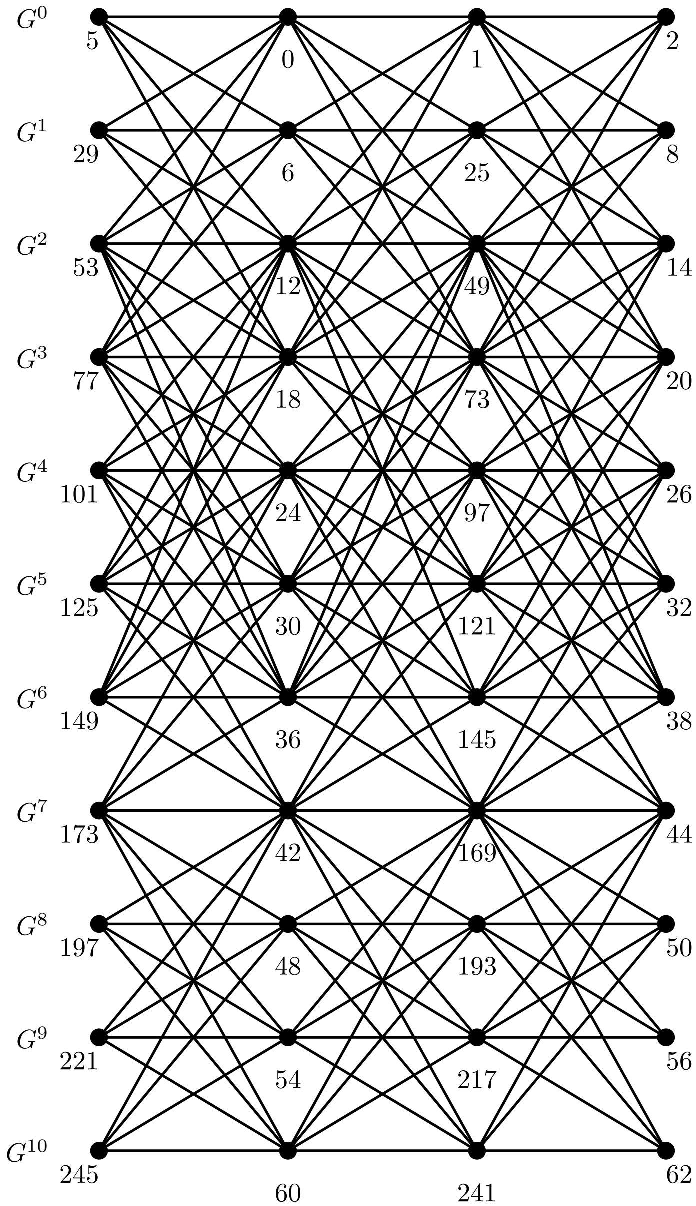


Figure 3: Odd harmonious labeling of $P_4 \bigtriangledown K(5)$

that $A = \{u_i \in V(G) : f(u_i) \leq \lambda\}$ and $B = \{u_i \in V(G) : f(u_i) > \lambda\}$. Let m^* be the number of edges in $G \bigtriangledown K(n)$. Consider the construction of $G \bigtriangledown K(n)$ as in Theorem 2.1. In $G \bigtriangledown K(n)$, denote the partite sets of copy G^i by A^i and B^i. For a vertex $u_j \in V(G)$, we denote the corresponding vertex in copy G^i by u_{ij}. Define the labeling function w on $V(G \bigtriangledown K(n))$ as follows:

$$w(u_{ij}) = \begin{cases} \lambda - f(u_j) + im & u_{ij} \in V_e^i, i = 0, 1, .., a-1 \\ m^* + \lambda + 1 - (f(u_j) + im(r_1+2)) & u_{ij} \in V_o^i \text{ and } i \in N_1 \\ m^* + \lambda + 1 - (w(u_{i-1j}) \\ \quad + m(r_t + r_{t+1} + 3 - |V_t \cap V_{t+1}|)) & u_{ij} \in V_o^i \text{ and } i \in N_2 \\ m^* + \lambda + 1 - (w(u_{i-1j}) \\ \quad + m(r_{t+1} + 2)) & u_{ij} \in V_o^i \text{ and } i \in N_3 \end{cases} \tag{14}$$

where $N_1 = \{0, 1, \ldots, r_1 + 1 - |V_1 \cap V_2|\}$, N_2 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}|) \right., \\ \left. \sum_{j=1}^{j=t} r_j + t - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|) \right]$$

for $t = 1, 2, \ldots, n-2$ and $|V_t \cap V_{t+1}| \neq 0$, and N_3 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|) \right., \\ \left. \sum_{j=1}^{j=t+1} r_j + t + 1 - (\sum_{j=1}^{j=t+1} |V_j \cap V_{j+1}|) \right]$$

for $t = 1, 2, \ldots, n-2$.

It is not difficult to prove that w is an $\alpha-$ labeling with characteristic $\lambda^* = \lambda + (a-1)m$, where a is the number of vertices in $K(n)$. □

Corollary 2.2.1. *If G has an $\alpha-$ labeling, then so as $G[\overline{K_r}], r \geq 2$.*

Proof. Let $G(n, m)$ be a bipartite graph with bipartition (A, B) and f be an α-labeling of G with characteristic λ. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ such that $A = \{u_i \in V(G) : f(u_i) \leq \lambda\}$ and $B = \{u_i \in V(G) : f(u_i) > \lambda\}$. Then the function $g : V(G[\overline{K}_r]) \rightarrow \{0, 1, 2, \ldots, r^2m\}$ defined by:

$$g(u_{ij}) = \begin{cases} \lambda - f(u_j) + im & u_{ij} \in A^i \\ r^2m + (\lambda + 1) - (f(u_j) + rim) & u_{ij} \in B^i \end{cases} \tag{15}$$

is an $\alpha-$ labeling with characteristic $\lambda^* = \lambda + m(r - 1)$. □

3 Odd graceful labeling

We note that if a graph G is odd harmonious then G is odd graceful. Since G is odd graceful, $V(G)$ can be partitioned into A, B, where

$$A = \{u \in V(G) : f(u) \text{ is even}\}$$

and

$$B = \{u \in V(G) : f(u) \text{ is odd}\}.$$

For the relation between odd graceful and strongly harmonious bipartite graphs, we mention to [18].

Proposition 3.0.1. *If a graph G is odd harmonious, then G is odd graceful.*

Proof. Let $G(n, m)$ be an odd harmonious graph with an odd harmonious labeling f. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and V_e, V_o be the partite sets of $V(G)$ such that $V_e = \{u_j \in V(G) : f(u_j) \text{ is even}\}$ and $V_o = \{u_j \in V(G) : f(u_j) \text{ is odd}\}$. Define $g : V(G) \rightarrow \{0, 1, 2, \ldots, 2m - 1\}$ by

$$g(u) = \begin{cases} f(u) & u \in A \\ 2m - f(u) & u \in B \end{cases} \tag{16}$$

It is not difficult to prove that g is an odd graceful labeling of G. □

Theorem 3.1. *If G is an odd graceful graph with an odd graceful labeling f such that G does not contain a path $P_3 = x_1x_2x_3$ and $f(x_1) < f(x_2) < f(x_3)$, then $G \bigtriangledown K(n)$ is odd graceful.*

Proof. Firstly, we will prove that if G is connected odd graceful graph with an odd graceful labeling f such that, G does not contain a path $P_3 = x_1x_2x_3$ and $f(x_1) < f(x_2) < f(x_3)$, then G is odd harmonious and the result follows by Theorem 2.1. It is clear that $f(u) < f(v)$, for each edge uv with $u \in A$ and $v \in B$, because of the continuity of G and that G must contain an edge uv such that $f(u) = 0$ and $f(v) = 2m - 1$. Define $g : V(G) \to \{0, 1, 2, \dots, 2m - 1\}$ by

$$g(u) = \begin{cases} f(u) & u \in A \\ 2m - f(u) & u \in B \end{cases} \tag{17}$$

Then g is odd harmonious labeling. Indeed, we have

(i) g is injective, since f is injective.

(ii) An edge $uv, u \in A$ and $v \in B$ has label $g(u) + g(v) = f(u) + 2m - f(v) = 2m - (f(v) - f(u))$.

Let $G(p, m)$ be a disconnected odd graceful graph and the above condition is satisfied for each component in G. Let $C_1, C_2, ...C_n$, be the components of G. We must have one of the following two cases:

Case 1. $f(u) < f(v)$, for each edge uv with $u \in A$ and $v \in B$, in all components and therefore G, again, is odd harmonious.

Case 2. $f(u) < f(v)$, for each edge uv with $u \in A$ and $v \in B$, in some components and $f(u) > f(v)$, for each edge uv with $u \in A$ and $v \in B$, in another components. Let $C_1, C_2, \dots, C_s$, be the components in which $f(u) < f(v)$, for each edge uv in $C_i, 1 \le i \le s$, with $u \in A$ and $v \in B$, and $C_{s+1}, C_{s+2}, \dots, C_n$, be the components in which $f(u) < f(v)$, for each edge uv in $C_i, s + 1 \le i \le n$, with $u \in A$ and $v \in B$. Note that $V(C_i)$ could be partitioned into $C_i^A = \{u \in C_i : u \in A\}$ and $C_i^B = \{u \in C_i : u \in B\}$. Consider the following partition of $V(G)$, $A^* = \{u : u \in C_i^A, 1 \le i \le s\} \cup \{u : u \in C_i^B, s + 1 \le i \le n\}$ and $B^* = \{u : u \in C_i^B, 1 \le i \le s\} \cup \{u : u \in C_i^A, s + 1 \le i \le n\}$.

Recall the construction of $G \bigtriangledown K(n)$ as in Theorem 2.1. Let

$$V(G) = \{u_1, u_2, \dots, u_p\}.$$

In $G \bigtriangledown K(n)$, let A^{*i} and B^{*i} be the partitions of copy G^i, corresponding to A^* and B^*. Let $\mid E(G \bigtriangledown K(n)) \mid = m^*$. Define the labeling $g : V(G) \rightarrow \{0, 1, 2, \ldots, 2m^* - 1\}$ by:

$$g(u_{ij}) = \begin{cases} f(u_j) + 2im & u_{ij} \in V_e^i, \text{ for } i = 0, 1, \cdots, a-1 \\ 2m^* - 2m + f(u_j) \\ \quad -2mi(r_1+2) & u_{ij} \in V_o^i \text{ and } i \in N_1 \\ g(u_{i-1j}) - 2m\,(r_t + r_{t+1} \\ \quad +3 - |V_t \cap V_{t+1}|) & u_{ij} \in V_o^i \text{ and } i \in N_2 \\ g(u_{i-1j}) - 2m\,(r_{t+1} + 2) & u_{ij} \in V_o^i \text{ and } i \in N_3, \end{cases} \tag{18}$$

where $N_1 = \{0, 1, \ldots, r_1 + 1 - |V_1 \cap V_2|\}$, N_2 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}|)\ , \right.$$
$$\left. \sum_{j=1}^{j=t} r_j + t - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|) \right]$$

for $t = 1, 2, \ldots, n-2$ and $|V_t \cap V_{t+1}| \neq 0$, and N_3 consists of integers in the interval

$$\left[\sum_{j=1}^{j=t} r_j + t + 1 - (\sum_{j=1}^{j=t} |V_j \cap V_{j+1}| - |V_t \cap V_{t+1}|)\ , \right.$$
$$\left. \sum_{j=1}^{j=t+1} r_j + t + 1 - (\sum_{j=1}^{j=t+1} |V_j \cap V_{j+1}|) \right]$$

for $t = 1, 2, \ldots, n-2$.

It is not difficult to show that g is an odd graceful labeling, See Figure 4. $\square$

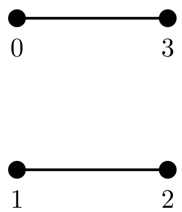
Corollary 3.1.1. *The following graphs are odd graceful:*

1. $nK_{m,m}$, *for all* $n, m \geq 2$.
2. $n(P_m \times P_2)$, *for all* $n, m \geq 2$.

Proof.

1. Note that $nK_{m,m} = nK_2 \bigtriangledown K_m = G \bigtriangledown K(2)$, where $K(2)$ is obtained form P_2 by joining u_1 and u_2 to every vertex in the complete graph K_{m-2}. Since nK_2 is odd graceful, the result follows immediately from Theorem 3.1. Note that, nK_2 has the odd graceful labeling $[0, 2n-1], [1, 2n-2], [2, 2n-3], \ldots, [n-1, n]$.

2. Since $n(P_m \times P_2) = nK_2 \bigtriangledown P_m$ and P_m belongs to $K(m)$, the result follows immediately. □



Odd graceful labeling of $2K_2$.

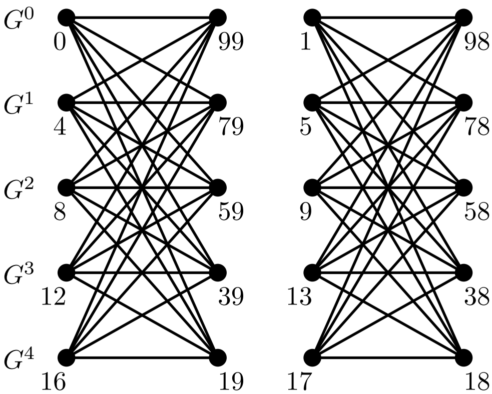


Figure 4: Odd graceful labeling of $2K_{5,5}$.

4 Odd harmonious labeling of the $m-$ splitting graph.

Recall that $G \bigtriangledown K_m$ can be constructed by taking m copies of G, say, $G^0, G^1, G^2, \ldots, G^{m-1}$ and join each vertex u in copy G^i to the neighbors of the corresponding vertex in copy G^j whenever y_i and y_j are adjacent in K_m. Therefore, $Spl_m(G)$ can be obtained from $G \bigtriangledown K_{m+1}$ by removing all edges in the copies G^i for all $i \geq 1$. Note that $Spl_m(G)$ has $|E(G)|(1+2m)$ edges, where $|E(G)|$ is the number of edges in G.

Theorem 4.1. *If G is an odd harmonious graph, then the graph $Spl_m(G)$, $m \geq 2$, is odd harmonious.*

Proof. Let $G(n, p)$ be an odd harmonious graph with odd harmonious labeling f. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and V_e, V_o be the partite sets of $V(G)$ such that $V_e = \{u_j \in V(G) : f(u_j) \text{ is even}\}$ and $V_o = \{u_j \in V(G) : f(u_j) \text{ is odd}\}$. Consider the construction of $Spl_m(G)$ as it is obtained from $G \bigtriangledown K_{m+1}$ by removing all the edges in G^i for all $i \geq 1$, as shown above.

In $Spl_m(G)$, denote the partite sets of copy G^i by V_e^i and V_o^i. For a vertex $u_j \in V(G)$, we denote the corresponding vertex in copy G^i by u_{ij}. Let g be the vertex labeling on $V(Spl_m(G))$ which is defined by:

$$g(u_{ij}) = \begin{cases} f(u_j) + 2ip & i = 0, 1, u_{ij} \in V_e^i \\ f(u_j) + 2p(2i-1) & i > 1, u_{ij} \in V_e^i \\ f(u_j) + 4ip & u_{ij} \in V_o^i \end{cases} \tag{19}$$

Then g is an odd harmonious labeling of $Spl_m(G)$. $\square$

Theorem 4.2. *If G has an $\alpha-$ labeling, then $Spl_m(G)$ has an $\alpha-$ labeling.*

Proof. Let $G(n,p)$ be a bipartite graph with bipartition (A,B) and let f be an α-labeling of G with characteristic λ. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ such that $A = \{u_i \in V(G) : f(u_i) \leq \lambda\}$ and $B = \{u_i \in V(G) : f(u_i) > \lambda\}$. Let m^* be the number of edges in $Spl_m(G)$. Consider the construction of $Spl_m(G)$ as in Theorem 4.1. In $Spl_m(G)$, denote the partite sets of copy G^i by A^i and B^i. For a vertex $u_j \in V(G)$, we denote the corresponding vertex in copy G^i by u_{ij}. Define the labeling function g on $V(Spl_m(G))$ as follows:

$$g(u_{ij}) = \begin{cases} f(u_j) + (m-i)p & u_{ij} \in A^i, i = 0, 1, 2, \ldots, m \\ m^* + (\lambda+1) - (f(u_j) + pi) & u_{ij}i, i = 0, 1, 2, \ldots, m \in B^i \end{cases} \tag{20}$$

g is an $\alpha-$ labeling of $Spl_m(G)$ with characteristic $\lambda^* = \lambda + mp$. $\square$

Theorem 4.3. *If G is an odd graceful graph with an odd graceful labeling f such that G does not contain a path $P_3 = x_1x_2x_3$ and $f(x_1) < f(x_2) < f(x_3)$, then $Spl_m(G)$ is odd graceful.*

Proof. Consider the construction of $G \bigtriangledown K(n)$ as in Theorem 3.1 and the construction of $Spl_m(G)$ as in Theorem 4.1. Let g be the vertex labeling on $V(Spl_m(G))$ which is defined by:

$$g(u_{ij}) = \begin{cases} f(u_j) + 2ip & u_{ij} \in A^{*i}, i = 0, 1 \\ f(u_j) + 2(2i-1)p & u_{ij} \in A^{*i}, i > 1 \\ 4pm + f(u_j) - 4ip & u_{ij} \in B^{*i} \end{cases} \tag{21}$$

In what follows, we will prove that g is injective and the odd gracefulness of it will follow immediately. To prove that g is injective, let $g(u_{ij}) = g(u_{kl})$, where $u_{ij} \in A^{*i}$ and $u_{kl} \in B^{*k}$.

For $i = 0$, we must have $f(u_i) = 4pm + f(u_l) - 4kp$. Hence $f(u_j) - f(u_l) = 4p(m-k)$ and we have the following cases :

1. If $m = k$, then $f(u_j) - f(u_l) = 0$ which is a contradiction.
2. If $m > k$, then $|f(u_j) - f(u_l)| > 4p$ which is a contradiction.

For $i = 1$, we must have $f(u_i) + 2p = 4pm + f(u_l) - 4kp$. Hence $f(u_j) - f(u_l) = p(4(m-k) - 2)$ and we have the following cases.

1. If $k = m$, then $|f(u_j) - f(u_l)| = 2p$ which is a contradiction.
2. If $k < m$, then $|f(u_j) - f(u_l)| > 2p$ which is a contradiction.

For $i \geq 2$, we must have $f(u_j) + 2(2i-1)p = 4pm + f(u_l) - 4kp$. Hence $f(u_j) - f(u_l) = p(4m + 2 - 4(i+k))$ and we have the following cases:

1. If $i + k = m$, then $f(u_j) - f(u_l) = 2p$ which is a contradiction.
2. If $i + k \neq m$, then $|f(u_j) - f(u_l)| > 2p$ which is a contradiction. $\square$

5 Further results

Let H^G be the graph obtained by identifying a distinguished vertex of G to each vertex of H. Snevily [16] showed that the graph formed by adding a pendant path P_n to each vertex of the cycle C_{4m} has an α-labeling. in the following theorem 5.2, we generalize this result by showing that C_{4m}^G is strongly odd harmonious when G is strongly odd harmonious. In [14], Seoud and Hafez proved that $T_n^{T_m}$ is strongly odd harmonious, where T_n and T_m are strongly odd harmonious trees on n and m vertices, respectively. The following result follows immediately, therefore we omitted the proof.

Theorem 5.1. *If G is a strongly odd harmonious graph and $H(n, n-1)$ is a strongly odd harmonious graph, then the graph H^G is strongly odd harmonious.*

Theorem 5.2. *If G is a strongly odd harmonious graph, then the graph C_{4m}^G is strongly odd harmonious.*

Proof. Let f be a strongly odd harmonious labeling of a graph $G(n,q)$. Denote the vertices of G by $x_0, x_1, \ldots, x_{n-1}$. Let G^1 and G^2 be two distinct copies of G. Denote the vertices of G^i by $x_{ij}, i = 1,2$ and $j = 0,1,2,\ldots,n-1$. Let x_j be an arbitrary fixed vertex in G and denote it by x_j^*. Let x_{ij}^* be the corresponding vertex to x_j^* in copy $G^i, i = 1,2$. Define $g(x_{1j}) = f(x_j)$ and $g(x_{2j}) = 2q+1-f(x_j)$, $j = 0,1,\ldots,n-1$. Now join the vertices x_{1j}^* and x_{2j}^* to get the new graph H with the strongly odd harmonious labeling g. Let $H^1, H^2, \ldots, H^{2m}$ be $2m$ distinct copies of H. Denote the vertices of H^r by x_{ij}^r, $i = 1,2$, $j = 0,1,2,\ldots,n-1$, and $r = 1,2,\ldots,2m$. Denote the vertices corresponding to x_{1j}^* and x_{2j}^* in copy H^r by x_{1j}^{r*} and x_{2j}^{r*} , $r = 1,2,\ldots,2m$. Label the vertices of H^r, $r = 1,2,3,\ldots,m$ by $g^\star(x_{ij}^r) = g(x_{ij}) + (r-1)(2q+2)$. Label the vertices of H^r, $r = m+1, m+2, \ldots, 2m$ by $g^\star(x_{ij}^r) = g(x_{ij}) + (2q+2)(r-1) + 2$ when $g(x_{ij})$ is even and $g^\star(x_{ij}^r) = g(x_{ij}) + (2q+2)(r-1)$ when $g(x_{ij})$ is odd. Now we join the vertex x_{1j}^{r*} to x_{2j}^{r+1*}, $r = 1,2,\ldots,2m-1$. Finally, we join the vertex x_{2j}^{1*} to x_{1j}^{2m*} to get the graph C_{4m}^G. The graph C_{4m}^G has $4m(q+1)$ edges. Since g is injective, $g^\star$ is injective. According the described labeling of C_{4m}^G, the maximum label assigned to any vertex is $2q + (2m-1)(2q+2) + 2 = 4m(q+1)$. Edges in H^r, $r = 1,2,\ldots,m$, have labels in the interval $[4(r-1)(q+1)+1, 4(r-1)(q+1)+4q+1]_o$. Edges in H^r, $r = m+1, m+2, \ldots, 2m$ have labels in the interval$[4(r-1)(q+1)+3, 4(r-1)(q+1)+4q+3]_o$. Edges $x_{1j}^{r*}x_{2j}^{r+1*}$, $r = 1,2,\ldots,m$, have labels $4(q+1)r-1$. Edges $x_{1j}^{r*}x_{2j}^{r+1*}$, $r = m+1, m+2, \ldots, 2m-1$ have labels $4(q+1)r+1$. Finally, the edge $x_{1j}^{2m*}x_{2j}^{1*}$ has label $4(q+1)m-1$. □

Lemma 5.2.1. *If a graph G has an α-labeling with characteristic $\lambda = \lfloor \frac{m}{2} \rfloor$, then G is strongly odd harmonious.*

Proof. Let G is a bipartite graph with bipartition (A,B) and f is an α-labeling of G such that $f(u) \leq \lambda = \lfloor \frac{m}{2} \rfloor$ for all $u \in A$, then the mapping 2

$$g(u) = \begin{cases} 2(\lambda - f(u)) & u \in A \\ 2(f(u) - \lambda) - 1 & u \in B \end{cases} \tag{22}$$

is a strongly odd harmonious labeling of G. Indeed, let m be even, without any loss of generality, then the vertices in A are assigned the labels in the set $\{0, 2, .., m\}$ and the vertices in B are assigned the labels in the set $\{1, 3, \ldots, m-1\}$. An edge uv with $u \in A$ and $v \in B$ is assigned the label $2\lambda - 2f(u) + 2f(v) - 2\lambda - 1 = 2(f(v) - f(u)) - 1$. Therefore edges are assigned the labels in the set $\{1, 3, \ldots, 2m-1\}$. □

In other words, a graph $G(n, m)$ is strongly odd harmonious if and only if G admits an $\alpha-$ labeling with characteristic $\lambda = \lfloor \frac{m}{2} \rfloor$.

Corollary 5.2.1. *If a graph G has an α-labeling with characteristic $\lambda = \lfloor \frac{m}{2} \rfloor$, then*

(1) *the graph P_n^G has an α-labeling, and*

(2) *the graph C_{4m}^G has an α-labeling.*

6 Conclusion and future work

In the present paper, we study some properties of odd harmonious labeling and the converse skew product to obtain more odd harmonious graphs. In [14], it is shown that if a graph G of size m has an odd harmonious labeling, then G decomposes both K_{2m+1} and $K_{m,m}$. General graphs are represented topologically by many authors as Nada et al. in [12]. Some others, for instance, Kozae et al. in [9] are interested to represent topological spaces by graphs. In the future, we investigate new approaches for graph decomposition and study their relation with rough set theory. The introduced techniques would be very useful in application because it opens the way for more topological applications from real-life problems.

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