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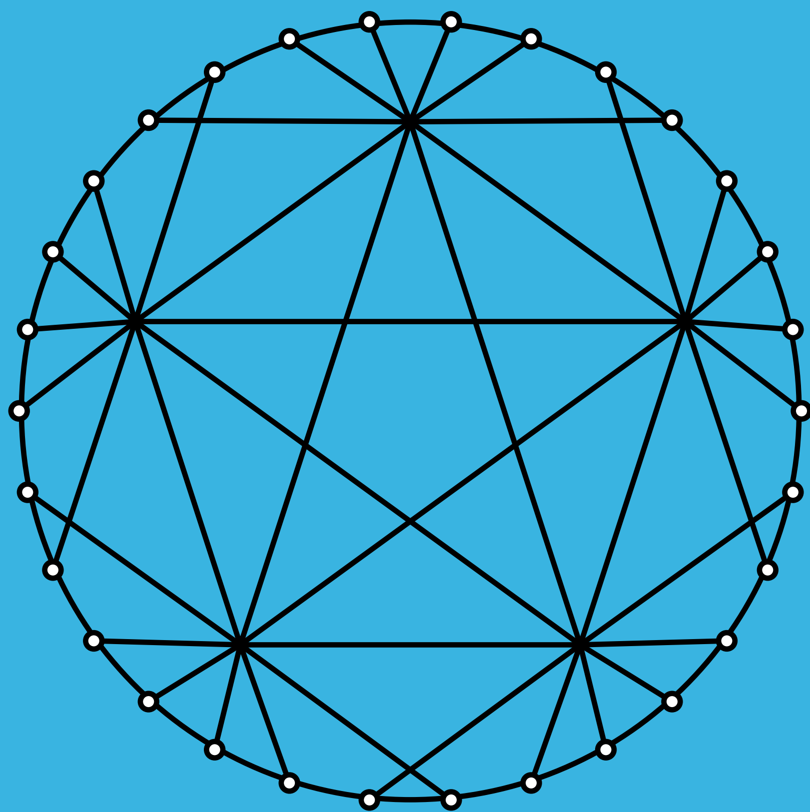

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On local antimagic chromatic number of a corona product graph

Gee-Choon Lau and Moviri Nalliah*

Abstract. In this paper, we provide a correct proof for the lower bounds of the local antimagic chromatic number of the corona product of friendship and fan graphs with null graph respectively as in [On local antimagic vertex coloring of corona products related to friendship and fan graph, *Indon. J. Combin.*, 5(2) (2021) 110–121]. Consequently, we obtained a sharp lower bound that gives the exact local antimagic chromatic number of the corona product of friendship and null graph.

1 Introduction

Let $G = (V, E)$ be a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by $|V(G)| = p$ and $|E(G)| = q$ respectively. The *friendship graph* f_n $(n \geq 2)$ is a graph which consists of n triangles with a common vertex. The *fan graph* F_n $(n \geq 2)$ is obtained by joining a new vertex to every vertex of a path P_n. The *corona product* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G along with $|V(G)|$ copies of H, and join the i-th vertex of G to every vertex of the i-th copy of H, where $1 \leq i \leq |V(G)|$. For integers $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$. For graph-theoretic terminology, we refer to Chartrand and Lesniak [4].

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Hartsfield and Ringel [7] introduced the concept of antimagic labeling of a graph. For a graph G, let $f : E(G) \to \{1, 2, ..., q\}$ be a bijection. For each vertex $u \in V(G)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. If $w(u) \neq w(v)$ for any two distinct vertices u and $v \in V(G)$, then f is called an antimagic labeling of G. Hartsfield and Ringel conjectured that every connected graph with at least three vertices admits antimagic labeling [7]. Interested readers can refer to [5, 6].

Arumugam et al. in [1], and independently, Bensmail et al. in [3], posed a new definition as a relaxation of the notion of antimagic labeling. They called a bijection $f : E \to \{1, 2, ..., |E|\}$ a *local antimagic labeling* of G if for any two adjacent vertices u and v in $V(G)$, the condition $w(u) \neq w(v)$ holds. Based on this notion, Arumugam et al. then introduced a new graph coloring parameter. Let f be a local antimagic labeling of a connected graph G. The assignment of $w(u)$ to u for each vertex $u \in V(G)$ induces naturally a proper vertex coloring of G which is called a *local antimagic vertex coloring* of G. The *local antimagic chromatic number*, denoted $\chi_{la}(G)$, is the minimum number of colors taken over all local antimagic colorings of G [1].

Arumugam et al. [2] obtained the local antimagic chromatic number for the graph $G \circ O_m$, where G is a path, cycle or complete graph and O_m is the null graph of order $m \geq 1$.

Theorem 1.1 (Arumugam et al. [2])**.** *Let $m \geq 2$, then*

$$\chi_{la}\left(C_3 \circ O_m\right) = 3m + 3,$$

except $\chi_{la}(C_3 \circ O_1) = 5$.

Theorem 1.2 (Arumugam et al. [2])**.** *For $n \geq 2$, $\chi_{la}\left(K_n \circ K_1\right) = 2n - 1$.*

In [8], the authors studied $\chi_{la}(f_n \circ O_m)$ and $\chi_{la}(F_n \circ O_m)$ for $n \geq 2$ and $m \geq 1$. We note that there are inconsistencies in the notations of f_n and F_n used. They proved that $\chi_{la}(f_n \circ O_m) \leq m(2n + 1) + 3$ and $\chi_{la}(F_n) \leq m(n + 1) + 3$ by providing a correct local antimagic labeling respectively. However, there are gaps in proving that $\chi_{la}(f_n \circ O_m) \geq m(2n + 1) + 3$ and $\chi_{la}(F_n) \geq m(n + 1) + 3$. Motivated by this, we shall first provide correct arguments to the proofs of the lower bounds. Consequently, we showed that $\chi_{la}(f_n \circ O_m) = m(2n + 1) + 2$ for $n \geq 2, m = 1$. Interested readers may refer to [9–12] for local antimagic chromatic number of graphs with pendant edges.

2 Lower bounds of $\chi_{la}(f_n \circ O_m)$ and $\chi_{la}(F_n \circ O_m)$

Lemma 2.1. *For $n \geq 2, m \geq 1$, $\chi_{la}(f_n \circ O_m) \geq m(2n+1)+3$ except $\chi_{la}(f_n \circ O_1) \geq m(2n+1)+2$.*

Proof. Let $G = f_n \circ O_m$ with $V(G) = \{x, u_i, v_i, x_j, u_j^i, v_j^i \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{xx_j, xu_i, xv_i, u_iv_i, u_iu_j^i, v_iv_j^i \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly, $|E(G)| = q = m(2n+1)+3n$.

Suppose $f : E(G) \to [1, q]$ is a local antimagic labeling of G. Clearly, all the $m(2n+1)$ pendant vertices must have distinct induced vertex colors that are at most q. Morever, $w(x) \geq 1+2+\cdots+(2n+m) = (2n+m)(2n+m+1)/2 = s$. Now, $2s - 2q = (2n+m+1)^2 + (2n+m+1) - 6n - 2m(2n+1) = 4n^2 + m^2 + m + 1 > 0$. Thus, $w(x) > q$. Therefore, $\chi_{la}(G) \geq m(2n+1)+1$. Without loss of generality, we consider the following 3 cases.

Case 1. $f(u_1v_1) = q$. In this case, $w(u_1) \neq w(v_1) \neq w(x) > q$ so that $\chi_{la}(G) \geq m(2n+1)+3$.

Case 2. $f(xu_1) = q$ or $f(u_1u_1^1) = q$. In this case, $w(u_1) \neq w(x) > q$ so that $\chi_{la}(G) \geq m(2n+1)+2$. Suppose equality holds. Clearly, for each $i \in [1, n]$, at most one of u_i, v_i has induced vertex color q. So, there are at most n vertices in $\{u_i, v_i\}$ with induced vertex color q. The sum of these n induced vertex colors is at least $1+2+\cdots+n(m+2) = n(m+2)[n(m+2)+1]$ and at most $nq = n[3n+m(n+1)]$. Since $n \geq 2$, it is easy to check that $n(m+2)[n(m+2)+1] - n[3n+m(n+1)] = 2n^2(m+1) + n + \frac{1}{2}mn(mn+1) - [3n^2 + mn(2n+1)] > 0$ if and only if $m > 1$. Consequently, $\chi_{la}(G) \geq m(n+1)+2$ if $m = 1$, and $\chi_{la}(G) \geq m(n+1)+3$ if $m \geq 2$.

Case 3. $f(xx_1) = q$. In this case, $w(x_1) = q$ and $w(u_j^i), w(v_j^i), w(x_j) < q$ $(x_j \neq x_1)$ so that $\chi_{la}(G) \geq m(2n+1)+1$. Suppose $w(v_i) < w(u_i) \leq q$ for $1 \leq i \leq n$, then $\sum_{i=1}^{n}[w(u_i)+w(v_i)]$ is at most $n(2q-1)$ and at least $1+2+\cdots+n(2m+3) = n(2m+3)[n(2m+3)+1]/2$. Now,

$$
\begin{aligned}
&n(2m+3)[n(2m+3)+1] - 2n(2q-1) \\
&\quad = n(2m+3)[n(2m+3)+1] - 2n[2m(2n+1)+6n-1] \\
&\quad = 4m^2n^2 + 4mn^2 - 2mn - 3n^2 + 5n > 0.
\end{aligned}
$$

Thus, we may assume $w(u_1) > q$. Since $w(u_1) \neq w(x)$, we have $\chi_{la}(G) \geq m(2n+1)+2$. Suppose equality holds. By an argument similar to that in Case 2, we have $\chi_{la}(G) \geq m(2n+1)+2$ if $m = 1$ and $\chi_{la}(G) \geq m(2n+1)+3$ if $m \geq 2$. □

Note that $F_2 \circ O_m = C_3 \circ O_m$, we next consider $F_n \circ O_m, n \geq 3, m \geq 1$.

Lemma 2.2. *For $n \geq 3, m \geq 1$, $\chi_{la}(F_n \circ O_m) \geq m(n+1)+3$.*

Proof. Let $G = F_n \circ O_m$ with $V(G) = \{x, x_j, v_i, v_j^i \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{xx_j, xv_i, v_iv_j^i \,|\, 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_iv_{i+1} \,|\, 1 \leq i \leq n-1\}$. Clearly, $|E(G)| = m(n+1)+2n-1 = q$.

Let f be a local antimagic labeling of G that induces $\chi_{la}(G)$ distinct vertex colors. Clearly, all the $m(n+1)$ pendant vertices must have distinct induced vertex colors that are at most q. Moreover, $w(x) \geq 1+2+\cdots+(m+n)(m+n+1)/2 = s$. Now $2s-2q = (m+n)(m+n+1)-2[m(n+1)+2n-1] = m^2-m+n^2-3n+1 > 0$ for $n \geq 3$. Thus, $w(x) > q$ and $\chi_{la}(G) \geq m(n+1)+1$. Without loss of generality, we consider the following cases.

Case 1. $f(v_1v_2) = q$ or $f(v_2v_3) = q$ if $n \geq 4$. In this case, $w(x) \neq w(v_1) \neq w(v_2) > q$. Thus, $\chi_{la}(G) \geq m(n+1)+3$.

Case 2. $f(xv_1) = q$ (or $f(xv_2) = q$). In this case, $w(x) \neq w(v_1) > q$ (or $w(x) \neq w(v_2) > q$). Thus, $\chi_{la}(G) \geq m(n+1)+2$. Suppose equality holds. Note that if $w(v_i) > q$ for $3 \leq i \leq n$, then $w(v_i) = w(v_1)$. Moreover, $w(v_i) \neq w(v_{i+1})$ for $1 \leq i \leq n-1$. Suppose there are $r \geq 1$ vertices in $\{v_i \,|\, 1 \leq 1 \leq n\}$ with induced vertex color larger than q, then there are $n-r \geq 1$ vertices in $\{v_i \,|\, 1 \leq 1 \leq n\}$ with induced vertex color at most q. These $n-r$ vertices are incident to a total of $(m+2)n-1-r(m+1) = (m+1)(n-r)+n-1$ edges. Therefore, their edge labels sum under f is at most $(n-r)q$. However, the sum is at least $S = 1+2+\cdots+[(m+1)(n-r)+n-1] = \frac{1}{2}[(m+1)(n-r)+n-1][(m+1)(n-r)+n]$. Note that $n-r \geq n/2$. Thus,

$$-r \geq -n/2 \qquad \text{and} \qquad 2S-2(n-r)q \geq \frac{n}{2}\left[\frac{m^2n}{2}+\frac{n}{2}-3\right] > 0$$

except for $n = 3, m = 1$. This contradicts $S \leq (n-r)q$ for all $(n,m) \neq (3,1)$.

The second inequality is obtained as follows:

$$\begin{aligned}
2S - 2(n-r)q = {} & [(m+1)(n-r)+n]^2 - [(m+1)(n-r)+n] \\
& \qquad - 2(n-r)[m(n+1)+2n] \\
&= (m+1)^2(n-r)^2 + (2n-1)(m+1)(n-r) \\
& \qquad + n^2 - n - 2(n-r)(mn+m+2n) \\
&= (n-r)[m^2(n-r) + 2m(n-r) - 3m - n - r - 1] + n^2 - n \\
&\geq (n-r)[m^2(n-r) + 2m(n-r) - 3m - \frac{3n}{2} - 1] + n^2 - n \\
&\geq \frac{n}{2}\left[(m^2+2m)(\frac{n}{2}) - 3m - \frac{3n}{2} - 1\right] + n^2 - n \\
&\geq \frac{n}{2}\left[\frac{m^2 n}{2} - \frac{3n}{2} - 1 + 2n - 2\right] \\
&= \frac{n}{2}\left[\frac{m^2 n}{2} + \frac{n}{2} - 3\right] > 0 \text{ except when } (n,m) = (3,1)
\end{aligned}$$

Now, consider $G = F_3 \circ O_1$ that has $q = 9$. If G admits a local antimagic labeling that induces 6 distinct vertex colors, then $w(v_1) = w(v_3) \leq 9$. Since v_1 and v_3 are incident to 6 different edges, their total label sum is at least 21 so that $w(v_1) = w(v_3) \geq 11$, a contradiction. Therefore, $\chi_{la}(G) \geq m(n+1)+3$.

Case 3. $f(v_1v_1^1) = q$ (or $f(v_2v_1^2) = q$). In this case, $w(v_1) \neq w(x) > q$ (or $w(v_2) \neq w(x) > q$). Thus, $\chi_{la}(G) \geq m(n+1)+2$. Suppose equality holds. By an argument similar to Case 2, we have the same contradiction. □

3 $\chi_{la}(f_n \circ O_1)$

In [8], the authors obtained local antimagic labelings that correctly show that $\chi_{la}(f_n \circ O_m) \leq m(2n+1)+3$ and $\chi_{la}(F_n \circ O_m) \leq m(n+1)+3$. By Lemma 2.1, we shall next show that $\chi_{la}(f_n \circ O_1) = 2n+3$.

Theorem 3.1. *For $n \geq 2$, $\chi_{la}(f_n \circ O_1) = 2n+3$.*

Proof. Let $G = f_n \circ O_1$ with $V(G)$ and $E(G)$ as defined in the proof of Lemma 2.1. Suffice to define a bijection $f : E(G) \to [1, 5n+1]$ that induces $2n+3$ distinct induced vertex colors. We shall use labeling matrices to describe the labeling of all the edges of $f_n \circ O_1$.

Suppose n is odd. We first define $f(xx_1) = 5n+1$. We now arrange integers in $[2n+1, 5n]$ as a $3 \times n$ matrix as follows:

(1) In row 1, assign $4n + (i+1)/2$ to column i if $i = 1, 3, 5 \ldots, n$; assign $(9n+1)/2 + i/2$ if $i = 2, 4, 6, \ldots, n-1$. We have used integers in $[4n+1, 5n]$.

(2) In row 2, assign $(7n+1)/2 + (i-1)/2$ to column i if $i = 1, 3, 5 \ldots, n$; assign $3n + i/2$ if $i = 2, 4, 6, \ldots, n-1$. We have used integers in $[3n+1, 4n]$.

(3) In row 3, assign $3n+1-i$ to column $1 \le i \le n$. We have used integers in $[2n+1, 3n]$.

The resulting matrix is given in Table 1.

Table 1: Assignment of integers in $[2n+2, 5n+1]$

$4n+1$	$\frac{9n+3}{2}$	$4n+2$	$\frac{9n+5}{2}$	$\cdots$	$5n-1$	$\frac{9n-1}{2}$	$5n$	$\frac{9n+1}{2}$
$\frac{7n+1}{2}$	$3n+1$	$\frac{7n+3}{2}$	$3n+2$	$\cdots$	$\frac{7n-3}{2}$	$4n-1$	$\frac{7n-1}{2}$	$4n$
$3n$	$3n-1$	$3n-2$	$3n-3$	$\cdots$	$2n+4$	$2n+3$	$2n+2$	$2n+1$

We next arrange integers in $[1, 3n]$ as a $3 \times n$ matrix as follows:

(1) In row 1, assign $3n+1-i$ to column $1 \le i \le n$. We have used integers in $[2n+1, 3n]$.

(2) In row 2, assign $(3n+1)/2 + (i-1)/2$ to column i if $i = 1, 3, 5, \ldots, n$; assign $n + i/2$ to column i if $i = 2, 4, 6, \ldots, n-1$. We have used integers in $[n+1, 2n]$.

(3) In row 3, assign $(i+1)/2$ to column i if $i = 1, 3, 5, \ldots, n$; assign $(n+1)/2 + i/2$ to column i if $i = 2, 4, 6, \ldots, n-1$. We have used integers in $[1, n]$.

The resulting matrix is given in Table 2.

Table 2: Assignment of integers in $[2n+2, 5n+1]$

$3n$	$3n-1$	$3n-2$	$3n-3$	$\cdots$	$2n+4$	$2n+3$	$2n+2$	$2n+1$
$\frac{3n+1}{2}$	$n+1$	$\frac{3n+3}{2}$	$n+2$	$\cdots$	$\frac{3n-3}{2}$	$2n-1$	$\frac{3n-1}{2}$	$2n$
1	$\frac{n+3}{2}$	2	$\frac{n+5}{2}$	$\cdots$	$n-1$	$\frac{n-1}{2}$	n	$\frac{n+1}{2}$

For $1 \le k \le 3$, $1 \le i \le n$, let $a_{k,i}$ be the (k,i)-entry of Table 1, and $b_{k,i}$ be the (k,i)-entry of Table 2. Note that $b_{1,i} = a_{3,i}$. Define $f(u_i u_1^i) = a_{1,i}$, $f(xu_i) = a_{2,i}$, $f(u_i v_i) = a_{3,i}$, $f(xv_i) = b_{2,i}$ and $f(v_i v_1^i) = b_{3,i}$. It is obvious that f is a bijective function.

Now, column sum of each column of Table 1 is $(21n+3)/2$. Thus, $w(u_i) = (21n+3)/2$ and $w(u_1^i) \in [4n+1, 5n]$ for $1 \le i \le n$. Similarly, the column sum of each column of Table 2 is $(9n+3)/2$. Thus, $w(v_i) = (9n+3)/2$ and $w(v_1^i) \in [1, n]$ for $1 \le i \le n$. Moreover, $w(x) = (n+1) + \cdots + (2n) + (3n+1) + \cdots + 4n + (5n+1) = (n+1)(5n+1)$. Clearly, $w(x) \neq w(u_1^i) \neq w(u_i) \neq w(v_1^i) \neq w(x_1) = 5n+1$ for $1 \le i \le n$. Note that $4n+1 \le w(v_i) = (9n+3)/2 \le 5n+1$ is odd for $n \ge 3$. Therefore, f is a local antimagic labeling that induces $2n+3$ distinct vertex colors. Consequently, $\chi_{la}(f_n \circ O_1) = 2n+3$ for odd $n \ge 3$.

We now consider even $n \ge 2$. Figures 1 and 2 show that $\chi_{la}(f_2 \circ O_1) = 7$ and $\chi_{la}(f_4 \circ O_1) = 11$.

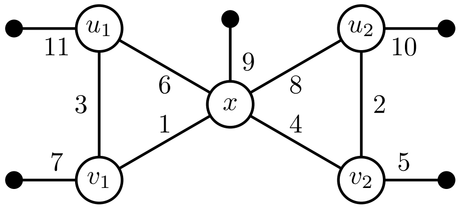

Figure 1: $\chi_{la}(f_2 \circ O_1)$=7 with induced vertex colors in $\{7, 5, 9, 10, 11, 20, 28\}$

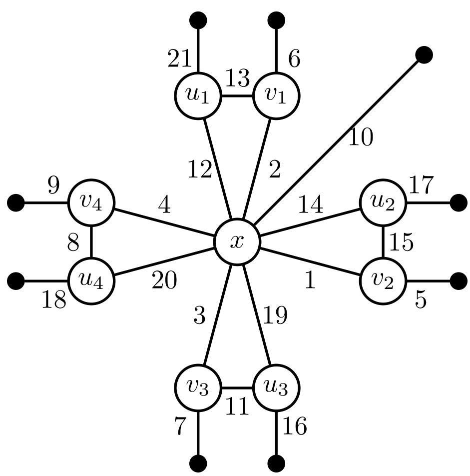

Figure 2: $\chi_{la}(f_4 \circ O_1)$=11 with induced vertex colors in $\{5, 6, 7, 9, 10, 16, 17, 18, 21, 46, 85\}$

Consider $n \geq 6$. We first define $f(xx_1) = 3n+3$, $f(u_nv_n) = 1$, $f(u_nu_1^n) = 2n+2$, $f(xu_n) = 2n$, $f(v_nv_1^n) = 2n+3$ and $f(xv_n) = 2n+1$. We now have $w(x_1) = 3n+3$, $w(u_n) = 4n+3$, $w(u_1^n) = 2n+2$, $w(v_n) = 4n+5$ and $w(v_1^n) = 2n+3$. We now consider the remaining integers in $[2, 2n-1] \cup [2n+4, 3n+2] \cup [3n+4, 5n+1]$.
We now arrange integers in $[2n+4, 3n+2] \cup [3n+4, 5n+1]$ as a $3 \times (n-1)$ matrix as follows:

(1) In row 1, assign $4n+3+(i-1)/2$ to column i if $i = 1, 3, 5 \ldots, n-1$; assign $9n/2+2+i/2$ if $i = 2, 4, 6, \ldots, n-2$. We have used integers in $[4n+3, 5n+1]$.

(2) In row 2, assign $7n/2+3+(i-1)/2$ to column i if $i = 1, 3, 5 \ldots, n-1$; assign $3n+3+i/2$ if $i = 2, 4, 6, \ldots, n-2$. We have used integers in $[3n+4, 4n+2]$.

(3) In row 3, assign $3n+3-i$ to column $1 \leq i \leq n-1$. We have used integers in $[2n+4, 3n+2]$.

The resulting matrix is given in Table 3.

Table 3: Assignment of integers in $[2n+4, 3n+2] \cup [3n+4, 5n+1]$

$4n+3$	$\frac{9n}{2}+3$	$4n+4$	$\frac{9n}{2}+4$	$\cdots$	$5n$	$\frac{9n}{2}+1$	$5n+1$	$\frac{9n}{2}+2$
$\frac{7n}{2}+3$	$3n+4$	$\frac{7n}{2}+4$	$3n+5$	$\cdots$	$\frac{7n}{2}+1$	$4n+1$	$\frac{7n}{2}+2$	$4n+2$
$3n+2$	$3n+1$	$3n$	$3n-1$	$\cdots$	$2n+7$	$2n+6$	$2n+5$	$2n+4$

We next arrange integers in $[2, 2n-1] \cup [2n+4, 3n+2]$ as a $3 \times n$ matrix as follows:

(1) In row 1, assign $3n+3-i$ to column $1 \leq i \leq n-1$. We have used integers in $[2n+4, 3n+2]$.

(2) In row 2, assign $3n/2+(i-1)/2$ to column i if $i = 1, 3, 5, \ldots, n-1$; assign $n+i/2$ to column i if $i = 2, 4, 6, \ldots, n-2$. We have used integers in $[n+1, 2n-1]$.

(3) In row 3, assign $(i+3)/2$ to column i if $i = 1, 3, 5, \ldots, n-1$; assign $n/2+1+i/2$ to column i if $i = 2, 4, 6, \ldots, n-2$. We have used integers in $[2, n]$.

The resulting matrix is given in Table 4.

Table 4: Assignment of integers in $[2, 2n-1] \cup [2n+4, 3n+2]$

$3n+2$	$3n+1$	$3n$	$3n-1$	$\cdots$	$2n+7$	$2n+6$	$2n+5$	$2n+4$
$\frac{3n}{2}$	$n+1$	$\frac{3n}{2}+1$	$n+2$	$\cdots$	$\frac{3n}{2}-2$	$2n-2$	$\frac{3n}{2}-1$	$2n-1$
2	$\frac{n}{2}+2$	3	$\frac{n}{2}+3$	$\cdots$	$n-1$	$\frac{n}{2}$	n	$\frac{n}{2}+1$

For $1 \le k \le 3$, $1 \le i \le n-1$, let $c_{k,i}$ be the (k,i)-entry of Table 3, and $d_{k,i}$ be the (k,i)-entry of Table 4. Note that $d_{1,i} = c_{3,i}$. Define $f(u_i u_1^i) = c_{1,i}$, $f(xu_i) = c_{2,i}$, $f(u_i v_i) = c_{3,i}$, $f(xv_i) = d_{2,i}$ and $f(v_i v_1^i) = d_{3,i}$. It is obvious that f is a bijective function.

Now, column sum of each column of Table 3 is $21n/2 + 8$. Thus, $w(u_i) = 21n/2 + 8$ and $w(u_1^i) \in [4n+3, 5n+1]$ for $1 \le i \le n-1$. Similarly, the column sum of each column of Table 4 is $9n/2+4$. Thus, $w(v_i) = 9n/2+4$ and $w(v_1^i) \in [2, n]$ for $1 \le i \le n-1$. Moreover, $w(x) = [2n + (2n+1) + (3n+3)] + (3n+4) + \cdots + (4n+2) + (n+1) + \cdots + (2n-1) = (7n+4) + (n-1)(5n+3) = 5n^2 + 5n + 1$. Clearly, for $1 \le i \le n-1$, $w(x) \ne w(u_1^i) \ne w(u_i) \ne w(v_1^i) \ne w(u_1^1) \ne w(v_1^1) \ne w(x_1)$. Note that $4n+3 \le w(u_n) = 4n+3 \ne w(v_n) = 4n+5 \le 5n+1$ for even $n \ge 6$. Therefore, f is a local antimagic labeling that induces $2n+3$ distinct vertex colors. Consequently, $\chi_{la}(f_n \circ O_1) = 2n+3$ for even $n \ge 6$. □

Example 3.1. Figures 3 and 4 below give the labelings of $f_3 \circ O_1$ and $f_6 \circ O_1$ according to the proof in Theorem 3.1.

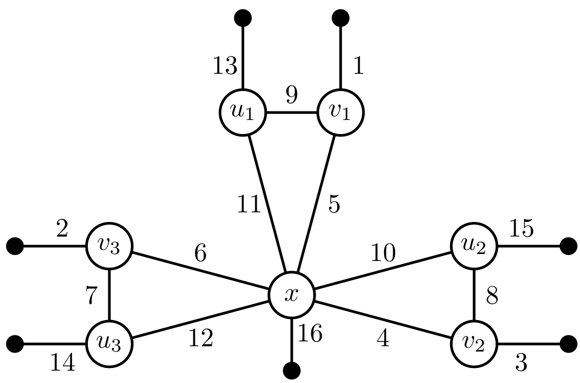

Figure 3: $\chi_{la}(f_3 \circ O_1) = 9$ with induced vertex colors in $[1, 3] \cup [13, 16] \cup \{33, 64\}$

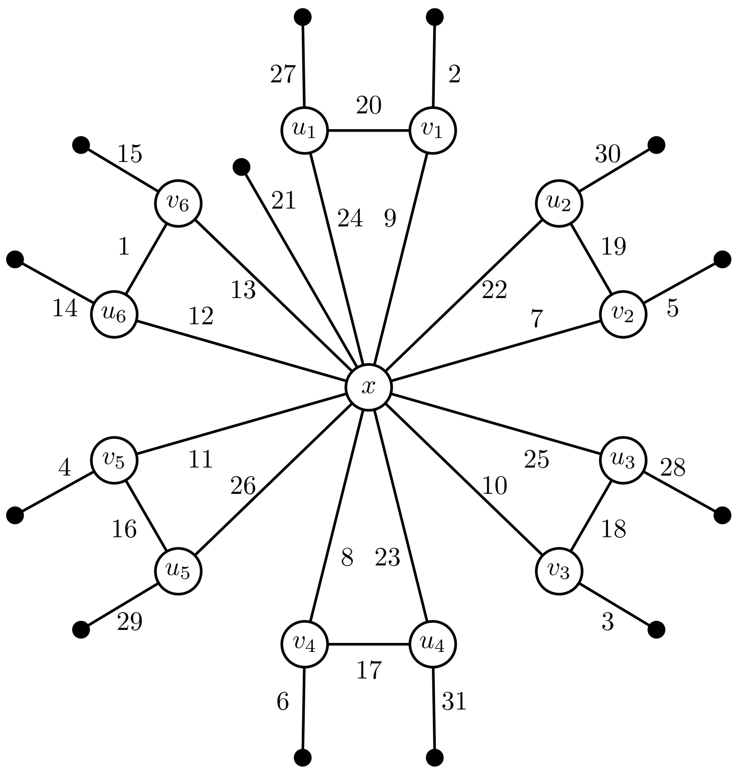

Figure 4: $\chi_{la}(f_6 \circ O_1) = 15$ with induced vertex colors in $[1, 6] \cup [27, 31] \cup \{21, 71, 211\}$

References

[1] S. Arumugam, K. Premalatha, M. Bača and A. Semaničová-Feňovčíková, Local Antimagic Vertex Coloring of a Graph, *Graphs Combin.*, **33** (2017),275–285.

[2] S. Arumugam, Y.-C. Lee, K. Premalatha, T.-M. Wang, On Local Antimagic Vertex Coloring for Corona Products of Graphs, https://arxiv.org/pdf/1808.04956.pdf

[3] J. Bensmail, M. Senhaji and K. Szabo Lyngsie, On a combination of the 1-2-3 conjecture and the antimagic labelling conjecture, *Discrete Math. Theor. Comput.Sci.*, **19(1)** (2017), No. 21.

[4] G. Chartrand and L. Lesniak, *Graphs & Digraphs (fourth edition)*, Chapman and Hall/CRC, 2005.

[5] T. Eccles, Graphs of large linear size are antimagic, *J. Graph Theory*, **81(3)** (2016), 236–261.

[6] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2019), DS6.

[7] N. Hartsfield and G. Ringel, *Pearls in graph theory*, Academic Press, (1994).

[8] Z.R. Himami and D.R. Silaban, On local antimagic vertex coloring of corona products related to friendship and fan graph, *Indon. J. Combin.*, **5(2)** (2021), 110–121.

[9] G.-C. Lau, H.-K. Ng and W.-C. Shiu, Affirmative Solutions On Local Antimagic Chromatic Number, *Graphs Combins*, **36(5)** (2020), 1337–1354.

[10] G.-C Lau, W.-C. Shiu and H.-K. Ng, On local antimagic chromatic number of graphs with cut-vertices, *Iran. J. Maths. Sci. Inform.*, (2022), to appear.

[11] G.-C Lau, W.-C. Shiu, and H.-K. Ng, On number of pendants in local antimagic chromatic numbers, *J. Discrete Math. Sci. Cryptogr.*, (2021).
DOI: 10.1080/09720529.2021.1920190.

[12] G.-C Lau, W.-C. Shiu, and C.-X. Soo, On local antimagic chromatic number of spider graphs, *J. Discrete Math. Sci. Cryptogr.*, (2022).
DOI : 10.1080/09720529.2021.1892270.

Gee-Choon Lau
College of computing, informatics & media, Universiti Teknologi MARA (Segamat Campus), 85000, Johor, Malaysia.
geeclau@yahoo.com

Moviri Nalliah
School of Advanced Sciences, Department of Mathematics, Vellore Institute of Technology, Vellore-632 014, India.
nalliahklu@gmail.com