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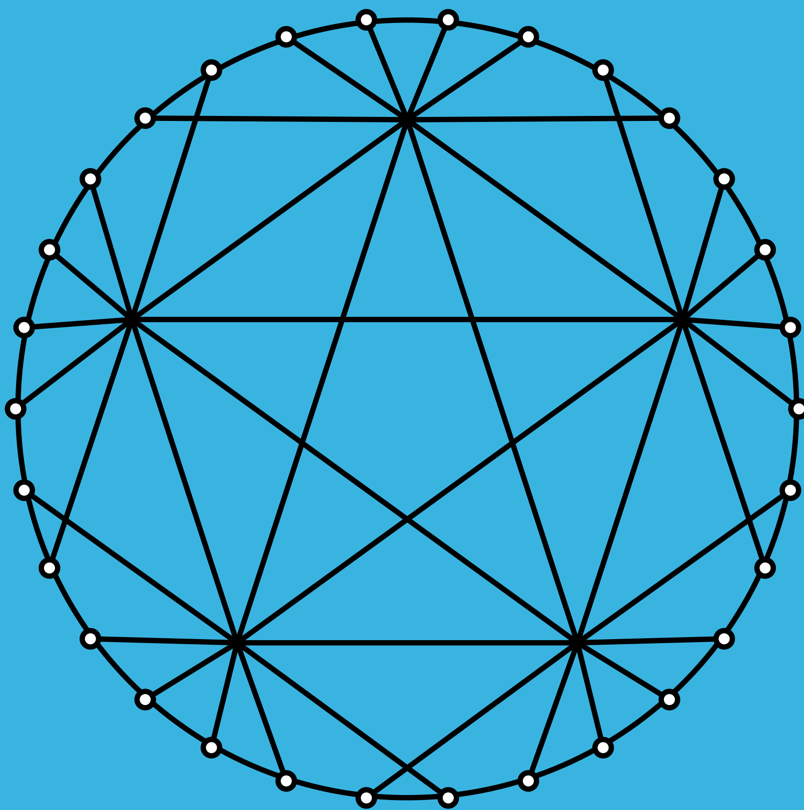

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A short note on Zeckendorf type numeration systems with negative digits allowed

Péter Hajnal*

Abstract. Several numeral systems are known, that are based on Fibonacci numbers. The best known is the Zeckendorf representation. Another one is the lazy Fibonacci representation of natural numbers. Both of them use 0-1 digits. A more recent one is due to Alpert. She allows negative digits in her representation. We introduce three more systems that use negative digits.

1 Introduction

Denote by $(F_i)_{i=0}^{\infty}$ the Fibonacci sequence ([12] A000045) defined as $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0$ and $F_1 = 1$. We refer to $F_2 < F_3 < F_4 < \cdots$ as the Fibonacci numbers. So when we mention Fibonacci numbers, we mean elements of the Fibonacci sequence with indices at least 2. Using the recursion formula one can extend the sequence to negative indices:

$$\ldots, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

Zeckendorf [15] introduced a numeral system based on Fibonacci numbers. As a new system of numeration is introduced, several natural mathematical problems are addressed. They are considering for example algorithmic problems of the arithmetic of numbers given by the Zeckendorf representation (for example [1], [14], [7]) and probabilistic questions on the distribution of the number of digit 1's among the numbers of given Zeckendorf length (for example [11], [9], [5]). The beauty of the system often comes with surprising connections. For example, the Zeckendorf representation provides the key to the complete analysis of a version of the Nim game [16].

*Corresponding author: hajnal@math.u-szeged.hu
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Other numerical systems were introduced, and they ignited similar lines of research. In Section 2 we give a short overview of previous notions. We assume familiarity with positional representations and the notation used therein. For the sake of self-containment we discuss the basics of the Zeckendorf representation. For a more detailed account of numerical systems see [13]. The first systems used non-negative digits, but later systems also used negative digits. We go further in this direction.

The main topic of this paper is to introduce new numeral systems. Our two main results are the following two theorems.

Theorem 1.1 (Alternating representation). *Every natural number n can be written uniquely as a sum*

$$n = F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \cdots + (-1)^{t-1}F_{i_{t-1}},$$

with $i_0 = \ell \gg i_1 \gg i_2 \gg \cdots \gg i_{t-3} \gg i_{t-2} \ggg i_{t-1} \geq 2$, where $i \gg j$ denotes $i > j+1$ (i.e. $i \geq j+2$) and $i \ggg j$ denotes $i > j+2$ (i.e. $i \geq j+3$). This sum is called the alternating representation of n.

The parameter t denotes the number of terms. Note that $n = F_\ell$ is an alternating representation, but $n = F_\ell - F_{\ell-2}$ is not (in the case $t = 2$ the requirement on the last two indices: $i_{t-2} \ggg i_{t-1}$ must be fulfilled).

Theorem 1.2 (Even representation). *Every natural number n can be written uniquely as a sum*

$$n = F_{2\ell} + \epsilon_{i_1}F_{2i_1} + \epsilon_{i_2}F_{2i_2} + \epsilon_{i_3}F_{2i_3} + \cdots + \epsilon_{t-1}F_{2i_{t-1}},$$

where $i_0 = \ell > i_1 > i_2 > \cdots > i_{t-1} \geq 1, \epsilon_i \in \{-1, 1\}$ and in the sequence $\epsilon_{i_0} = 1, \epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_{t-1}}$ there are no two consecutive -1's. This sum is called the even representation of n.

Theorem 1.1 and Theorem 1.2 are proven in Section 3 and Section 4, respectively. We finish the paper with Section 5, where we discuss the relations between the classical and new theorems. We also introduce a third numeral system for natural numbers, that is call odd representation.

Throughout the paper the set $\{0, 1, 2, 3, \ldots\}$, i.e. the set of natural numbers is denoted as $\mathbb{N}$. The set of positive integers is denoted by $\mathbb{N}_+$. For i and j, two natural numbers $i \gg j$ denotes $i > j + 1$ (i.e. $i \geq j + 2$) and $i \ggg j$ denotes $i > j + 2$ (i.e. $i \geq j + 3$). The intervals are always intervals of $\mathbb{Z}$, so $]2, 6] = (2, 6] = \{3, 4, 5, 6\}$, $[2, 2] = \{2\}$. $A\dot{\cup}B$ denotes $A \cup B$ and contains the additional information that A and B are disjoint. $A\dot{\cup}B\dot{\cup}C\dot{\cup}$ denotes $A \cup B \cup C \cup \cdots$ and contains the additional information that $A, B, C, \ldots$ are pairwise disjoint.

2 Representations of natural numbers based on Fibonacci numbers

Every natural number can be written as a sum of Fibonacci numbers. The claim is true since $n = n \cdot F_2 = F_2 + F_2 + \cdots + F_2$ (0 is considered as an empty sum).

Observation 2.1. *Every natural number can be written as a sum of distinct Fibonacci numbers.*

To see the claim consider an arbitrary positive integer n and take all the possible terms $1 < 2 < 3 < 5 < \cdots < F_{\ell-1} < F_\ell$ $(\leq n < F_{\ell+1})$. We sketch two possible strategies/algorithms to come up with a representation:

(1) Eager or Greedy strategy/rule: Take F_ℓ as the first term and try to complete the representation, continuing with $n - F_\ell$.

(2) Lazy strategy/rule: If the sum of the numbers $F_{\ell-1} > \cdots > 5 > 3 > 2 > 1$ is at least n we throw away F_ℓ and try to recursively complete the representation. If the sum is smaller than n, we are forced to take F_ℓ as the first term and proceed with $n - F_\ell$.

When we have a representation of n as in Observation 2.1 we can code n using place-value notation. The places/positions correspond to the Fibonacci numbers: $\ldots, 8, 5, 3, 2, 1$ and the Fibonacci digits are $0, 1$. For example,

$$\begin{aligned} 2021 &= F_{17} + F_{14} + F_9 + F_7 = F_{16} + F_{15} + F_{14} + F_9 + F_7 \\ &= F_{16} + F_{15} + F_{13} + F_{11} + F_{10} + F_8 + F_7 + F_6 + F_5 + F_4 + F_3 + F_2 \end{aligned}$$

can be coded as:

$$2021 = 1001000010100000_F = 111000010100000_F = 110101101111111_F.$$

In this paper, $\ell(n)$ denotes the maximum index in the sum with respect to the representation of n.

Two representations play a central role in further research.

Theorem 2.2 (Brown [3])**.** *Every natural number has a unique representation as a sum of Fibonacci numbers such that except for the highest indexed term no two consecutive Fibonacci numbers are missing.*

Theorem 2.3 (Zeckendorf [15]). *Every natural number has a unique representation as a sum of nonconsecutive Fibonacci numbers.*

The existence in Theorem 2.2 can be proven by applying the Lazy rule repeatedly. The proof of uniqueness is standard (for example see [3]). The representation is called the lazy Fibonacci representation. The existence in Theorem 2.3 can be proven by applying the Eager rule repeatedly. The proof of uniqueness is standard (for example see [15]). The representation is called the Zeckendorf representation.

Sometimes these representations are called dense, resp. sparse representations.

To emphasize the two special forms we use Z, ℓF subscripts in the case of Zeckendorf, resp. lazy Fibonacci representation:

$$2021 = 1001000010100000_Z = 110101101111111_{\ell F}.$$

In 1992 Bunder [4] introduced a Zeckendorf type representation using Fibonacci numbers with negative indices. He proved that every integer has a unique representation as a sum of nonconsecutive Fibonacci numbers with negative indices. Using $F_{-i} = (-1)^{i+1}F_i$, Bunder's theorem can be rewritten as follows.

Theorem 2.4 (Bunder [4]). *Every integer number n can be written uniquely as a sum*

$$n = \epsilon_{i_0}F_{i_0} + \epsilon_{i_1}F_{i_1} + \epsilon_{i_2}F_{i_2} + \epsilon_{i_3}F_{i_3} + \cdots + \epsilon_{t-1}F_{i_{t-1}},$$

where $i_0 \gg i_1 \gg i_2 \gg \cdots \gg i_{t-1} \geq 1$ and $\epsilon_{i_j} = 1$, when i_j is odd, $\epsilon_{i_j} = -1$, when i_j is even.

In 2009 Alpert [2] created a new way to represent natural numbers (in fact Theorem 2.5 can be extended to work on integers), that will be important for us. She also uses Fibonacci numbers as places, but the digits are $-1, 0, 1$ (as in the case of our interpretation of Bunder's theorem).

Theorem 2.5 (Alpert [2]). *Every natural number n can be written uniquely as a sum*

$$n = F_\ell + \epsilon_{i_1}F_{i_1} + \epsilon_{i_2}F_{i_2} + \epsilon_{i_3}F_{i_3} + \cdots + \epsilon_{t-1}F_{i_{t-1}},$$

where $i_0 = \ell > i_1 > i_2 > \cdots > i_{t-1} \geq 2$ and $\epsilon_{i_0} = 1, \epsilon_{i_1}, \epsilon_{i_2}, \dots \epsilon_{t-1} \in \{-1, 1\}$. Furthermore if $\epsilon_{i_j} = \epsilon_{i_{j+1}}$, then $i_j - i_{j+1} \geq 4$; if $\epsilon_{i_j} = -\epsilon_{i_{j+1}}$, then $i_j - i_{j+1} \geq 3$.

The parameter t, in the theorem, is the number of terms in the representation. Note that $t = 0$ and $t = 1$ are possible. Hence 0 and the Fibonacci numbers can be represented as stated in the theorem. We will denote the -1 digit as $\bar{1}$ and we use subscript A when representing n using place-value notation.

For example,

$$\begin{aligned} 2021 &= 2584 - 610 + 55 - 8 \\ &= F_{18} - F_{15} + F_{10} - F_6 \\ &= 100\bar{1}0000100\bar{1}0000_A. \end{aligned}$$

3 Alternating representation, proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Assume

$$n = F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \cdots + (-1)^{t-1} F_{i_{t-1}},$$

$\ell \gg i_1 \gg \cdots \gg i_{t-2} \ggg i_{t-1} \geq 2$.

Observation 3.1. $n \leq F_\ell$.

Indeed, the first term is F_ℓ, the remainder terms can be paired:

$$n = F_\ell - (F_{i_1} - F_{i_2}) - (F_{i_3} - F_{i_4}) - \cdots .$$

All pairs (a sum of two signed Fibonacci numbers) and the possible last term are negative. Hence $n \leq F_\ell$.

Observation 3.2. $n \geq F_\ell - F_{\ell-2} + 1 = F_{\ell-1} + 1$.

The cases $\ell = 2, 3, 4$ easy, since $n = F_\ell$ is the only choice. If $t = 1$, the estimate is correct. If $t = 2$ ($n = F_\ell - F_{i_1}$), then $\ell \ggg i_1$ and the bound is true. If $t \geq 3$, i.e. we have at least three terms then we can pair the terms after:

$$n = F_\ell - F_{i_1} + (F_{i_2} - F_{i_3}) + (F_{i_4} - F_{i_5}) + \cdots .$$

All pairs and the possible last term are positive. Hence $n > F_\ell - F_{i_1} \geq F_\ell - F_{\ell-2}$.

$\mathbb{N}_+$ is the disjoint union of all intervals of the form $(F_{\ell-1}, F_\ell]$, i.e. $n \in (F_{\ell-1}, F_\ell]$ and we have that ℓ is unique. The two observations give us that if n can be written as required, then the first term of the representation can be determined.

Lemma 3.3. *Each natural number n has an alternating representation.*

Proof. We use induction on n. For $n = 0, 1, 2, 3$ the claim can be checked. Assume that $n \in [F_{\ell-1} + 1, F_\ell]$ ($\ell \geq 5$). Consider $F_\ell - n$. We know that $0 \leq F_\ell - n \leq F_\ell - (F_{\ell-1} + 1) = F_{\ell-2} - 1$. We distinguish two cases.

Case 1: $0 \leq F_\ell - n \leq F_{\ell-3}$. Then by the induction hypothesis, we know that the representation of $F_\ell - n$ exists and its first term is F_k, where k is at most $\ell - 3$. Rearranging the equality we obtain a representation of n.

Case 2: $F_{\ell-3} + 1 \leq F_\ell - n \leq F_{\ell-2} - 1$. Again by the induction hypothesis we know that the representation of $F_\ell - n$ exists and its first term must be $F_{\ell-2}$. Furthermore, we also know that the number of terms is at least two. The end of the proof of existence is as above. □

Lemma 3.4. *For each natural number n the alternating representation is unique.*

Proof. We use induction on n. For small values the claim is obvious. 0 can be represented as of the empty sum and that is the only way to do (Observation 3.2). For the induction step assume that $n \in [F_{\ell-1}+1, F_\ell]$. We prove by contradiction. Assume that we have two different representations. Since the first term of both representations are F_ℓ, we have

$$\begin{aligned}
n &= F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \cdots + (-1)^{t-1} F_{i_{t-1}} \\
&= F_\ell - F_{j_1} + F_{j_2} - F_{j_3} + \cdots + (-1)^{s-1} F_{j_{s-1}}, \\
F_\ell - n &= F_{i_1} - F_{i_2} + F_{i_3} - \cdots + (-1)^{t-2} F_{i_{t-1}} \\
&= F_{j_1} - F_{j_2} + F_{j_3} - \cdots + (-1)^{s-2} F_{j_{s-1}}.
\end{aligned}$$

We obtained two different representations of $F_\ell - n$ that is a natural number smaller than n: $F_\ell - n \leq F_\ell - (F_{\ell-1} + 1) \leq F_{\ell-2} - 1 < n$. That is a contradiction with the induction hypothesis. □

Lemma 3.3 and Lemma 3.4 imply Theorem 1.1.

An algorithm for finding the representation of n can be easily deduced from the proof: If $n = 0$, the representation is the empty sum. The first term F_ℓ can be determined from Observation 3.1 and 3.2. Then apply the algorithm recursively for $F_\ell - n$. Take this representation and then subtract (flip the 1's and -1's) in the representation of and add the representations digit wise to F_ℓ.

Theorem 1.1 (existence and uniqueness) leads to a new numeration system, where the positions correspond to Fibonacci numbers and the digits are $\bar{1} \equiv -1$, 0 and 1. We use the subscript *Alt* to denote that we use this representation of numbers. We finish the section with examples

$$\begin{array}{llll} 1 = 1_{Alt} & 2 = 10_{Alt} & 3 = 100_{Alt} & 4 = 100\bar{1}_{Alt} \\ 5 = 1000_{Alt} & 6 = 100\bar{1}0_{Alt} & 7 = 1000\bar{1}_{Alt} & 8 = 10000_{Alt} \\ 9 = 10\bar{1}001_{Alt} & 10 = 100\bar{1}00_{Alt} & 11 = 1000\bar{1}0_{Alt}. & \end{array}$$

4 Even representation, proof of Theorem 1.2

Let

$$n = F_{2\ell} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \cdots + \epsilon_{t-1} F_{2i_{t-1}}, \tag{1}$$

satisfying $\epsilon_{i_j} \in \{-1, 1\}$ and there are no two consecutive -1's in the sequence of ϵ's.

Observation 4.1. $n \leq F_{2\ell} + F_{2\ell-2} + F_{2\ell-4} + \cdots + F_2 = F_{2\ell+1} - 1.$

We can upper bound n with the sum of all possible positive terms. The last equality is well-known (see [10], Corollary 5.1, page 83).

Observation 4.2. $n \geq F_{2\ell} - F_{2\ell-2} = F_{2\ell-1}.$

Pair each negative term with the previous one (a necessarily positive term). We put each difference into one bracket, and consider the brackets as one term of a sum. With this view, the first term in the representation of n is at least $F_{2\ell} - F_{2\ell-2}$, the further terms are positive. The observation is proven.

$\mathbb{N}_+$ is the disjoint union of all intervals of the form $[F_{2\ell-1}, F_{2\ell+1})$, i.e. $n \in [F_{2\ell-1}, F_{2\ell+1})$ and we have that ℓ is unique, i.e. if n can be written as required, then ℓ can be determined.

Existence: We use induction on n. For $n = 0, 1, 2, 3$ the claim is obvious. Assume that $n \in [F_{2\ell-1}, F_{2\ell+1} - 1]$ ($\ell \geq 2$). We distinguish two cases for proving the induction step.

Case 1: $n \in [F_{2\ell}, F_{2\ell+1} - 1]$. Write n as $F_{2\ell} + n'$. Note that $0 \leq n' \leq F_{2\ell-1} - 1 (< n)$. Hence by induction n' has a representation that starts with F_{2i}, where $n' \in [F_{2i-1}, F_{2i+1})$. We know that $n' \leq F_{2\ell-1} - 1$, so $i < \ell$. The representation of n' give us the representation of n.

Case 2: $n \in [F_{2\ell-1}, F_{2\ell} - 1]$. Write n as $F_{2\ell} - F_{2\ell-2} + n'$. Note that $0 \leq n' \leq F_{2\ell-2} - 1 (< n)$. Similar to Case 1 n' has a representation that starts with F_{2i}, where $i < \ell$:

$$n' = F_{2i} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \cdots + \epsilon_{t-1} F_{2i_{t-1}}.$$

Hence

$$n = F_{2\ell} - F_{2\ell-2} + F_{2i} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \cdots + \epsilon_{t-1} F_{2i_{t-1}}.$$

Either $2i = 2\ell - 2$ or $2i < 2\ell - 2$. We are done in both cases.

The two cases cover all possibilities, hence the proof of existence is complete.

Uniqueness: We use induction on n. For small values the claim is obvious. For the induction step assume that $n \in [F_{2\ell-1}, F_{2\ell+1} - 1]$. We prove by contradiction. Again we distinguish two cases:

Case 1: $n \in [F_{2\ell}, F_{2\ell+1} - 1]$. Similar to Lemma 3.4 write n as $F_{2\ell} + n'$. Both representations of n start with $F_{2\ell}$ and is followed with a representation of n'. This leads to two different representations of $n' (< n)$. This contradicts the hypothesis of the induction step.

Case 2: $n \in [F_{2\ell-1}, F_{2\ell} - 1]$.

$$\begin{aligned} n =& F_{2\ell} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \cdots + \epsilon_{t-1} F_{2i_{t-1}} \\ =& F_{2\ell} + \epsilon'_{i'_1} F_{2i'_1} + \epsilon'_{i'_2} F_{2i'_2} + \epsilon'_{i'_3} F_{2i'_3} + \cdots + \epsilon'_{t-1} F_{2i'_{t-1}}. \end{aligned}$$

We can conclude that

$$\begin{aligned} n - F_{2\ell} + F_{2\ell-2} =& F_{2\ell-2} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \cdots + \epsilon_{t-1} F_{2i_{t-1}} \\ =& F_{2\ell-2} + \epsilon'_{i'_1} F_{2i'_1} + \epsilon'_{i'_2} F_{2i'_2} + \cdots + \epsilon'_{t-1} F_{2i'_{t-1}}. \end{aligned}$$

We see two different representations of $n - F_{2\ell} + F_{2\ell-2}(< n)$, a contradiction.

The two cases cover all possibilities, hence the proof of uniqueness is complete.

This finishes the proof of Theorem 1.2.

The interested reader may formulate a simple, efficient algorithm for determining the even representation of a given n that follows the logic of the proof.

The Theorem 1.2 (existence and uniqueness) leads to a new numeration system, where the positions correspond to Fibonacci numbers with even indices $(\dots, 121, 55, 21, 8, 3, 1)$ and the digits are $\bar{1} \equiv -1$, 0 and 1. We use the subscript *Even* when we use this representation of numbers. We finish the section with examples

$$\begin{array}{rrrr} 1 = 1_{Even}, & 2 = 1\bar{1}_{Even}, & 3 = 10_{Even}, & 4 = 11_{Even}, \\ 5 = 1\bar{1}0_{Even}, & 6 = 1\bar{1}1_{Even}, & 7 = 10\bar{1}_{Even}, & 8 = 100_{Even}, \\ 9 = 101_{Even}, & 10 = 11\bar{1}_{Even}, & 11 = 110_{Even}, & 12 = 111_{Even}, \\ 13 = 1\bar{1}00_{Even}, & 14 = 1\bar{1}01_{Even}, & 15 = 1\bar{1}1\bar{1}_{Even}, & 16 = 1\bar{1}10_{Even}. \end{array}$$

5 Relations between different Zeckendorf type numeration systems

In this section, we look at the relationships between different Zeckendorf type numeration systems. So far we have not assumed knowledge of any other (the original or Alpert's one) Zeckendorf type numeration system in our proofs. However, many readers might be familiar with the Zeckendorf representation. In this case, the question of finding a representation of a given number n may arise, starting from the Zeckendorf representation. We can also think the other way round. We can derive the Zeckendorf numeral system from one of the introduced new representations.

Perhaps the simplest such idea is the rewriting of even representations. It was implied in the proof that the even representation of an arbitrary natural number n can be conceived of as a sum, in which the terms are in the form of F_{2i} and $(F_{2j} - F_{2k})$. The second type of terms can subsequently be rewritten by the following well-known identity:

$$F_{2j} - F_{2k} = F_{2j-1} + F_{2j-3} + \cdots + F_{2k+1},$$

assuming $j > k$. After the transformation we obtain a Zeckendorf representation of n.

This insight can be reversed. Represent n using the Zeckendorf numeral system. Extract the Fibonacci numbers with odd indices and divide them into maximal segments containing consecutive odd indices. These segments can be summed based on the above identity and replaced by terms of the form $F_{2i} - F_{2j}$. We have thus sketched a new proof of the existence of an even representation based on the knowledge of the Zeckendorf system.

Uniqueness can be discussed similarly.

Note that the Alpert's and our alternating systems can be relatively easily rewritten into each other. Consider the Alpert representation of n. If the signs alternate, we stop. If they don't alternate, take the first two successive like-signed terms. We assume that they are both positive

$$n = \cdots - F_s + F_i + F_j \pm F_t + \cdots$$

where $s \geq i + 3$ and $i \geq j + 4$. Then, we can rewrite the identity apply the basic Fibonacci recursion:

$$n = \cdots - F_s + F_{i+1} - F_i + F_j \pm F_t + \cdots .$$

Similar rewriting is possible when the first two successive like-signed terms are both negative. We can continue the process for the next pair of successive like signs until we are done. We have constructed the alternative representation of n. This process is reversible. So if the alternate representation of n is given, we could then create Alpert's representation.

We have not followed this path. We always reasoned from the ground up, without reference to other systems. For many people the above rewriting ideas are more natural than the proofs we presented. Thinking them through is a productive work. For example, the idea of rewriting the Zeckendorf representation to an even representation, and the well-known identity

$$F_{2j+1} - F_{2k+1} = F_{2j} + F_{2j-2} + \cdots + F_{2k+2},$$

assuming $j > k$, lead to the introduction of an odd representation.

Theorem 5.1 (Odd representation)**.** *Every natural number n can be written uniquely as a sum*

$$n = F_{2\ell+1} + \epsilon_{i_1} F_{2i_1+1} + \epsilon_{i_2} F_{2i_2+1} + \epsilon_{i_3} F_{2i_3+1} + \cdots + \epsilon_{t-1} F_{2i_{t-1}+1} + \epsilon,$$

where $i_0 = \ell > i_1 > i_2 > \cdots > i_{t-1} \geq 1$,$\epsilon_i \in \{-1, 1\}$, $\epsilon \in \{0, -1\}$ *and in the sequence* $\epsilon_{i_0} = 1, \epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_{t-1}}, \epsilon$ *there are no two consecutive* -1*'s. This sum is called the odd representation of* n*.*

The "rewriting proof" and "decoding the first term and do induction" are two straightforward ways to prove the theorem. The exact argument is left to the reader.

In the case of odd representation we can use place-value notation again. The digits are $1, 0, \bar{1}$, the places are the odd indexed Fibonacci numbers $\ldots, F_7 = 13, F_5 = 5, F_3 = 2$. In the case of $\epsilon = 0$ we do not need anything more. In the case of $\epsilon = -1$ we finish the place-value code with ".$\bar{1}$". We use the subscript *Odd* to denote that we use this representation of numbers.

We finish the section with examples

$$\begin{array}{llll} 1 = 1.\bar{1}_{Odd}, & 2 = 1_{Odd}, & 3 = 1\bar{1}_{Odd}, & 4 = 10.\bar{1}_{Odd}, \\ 5 = 10_{Odd}, & 6 = 11.\bar{1}_{Odd}, & 7 = 11_{Odd}, & 8 = 1\bar{1}0_{Odd}, \\ 9 = 1\bar{1}1.\bar{1}_{Odd}, & 10 = 1\bar{1}1_{Odd}, & 11 = 10\bar{1}_{Odd}, & 12 = 100.\bar{1}_{Odd}, \\ 13 = 100_{Odd}, & 14 = 101.\bar{1}_{Odd}., & 15 = 101_{Odd}, & 16 = 11\bar{1}_{Odd}. \end{array}$$

6 Conclusion

We have presented three numeral systems based on Fibonacci numbers. There are not too many of these. They are natural systems but one must recognize their existence. We hope that these natural and beautiful systems will stimulate further research. The discussions of algorithmic, probabilistic, and computational issues require new techniques and ideas.

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Péter Hajnal
Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, Hungary 6720
hajnal@math.u-szeged.hu