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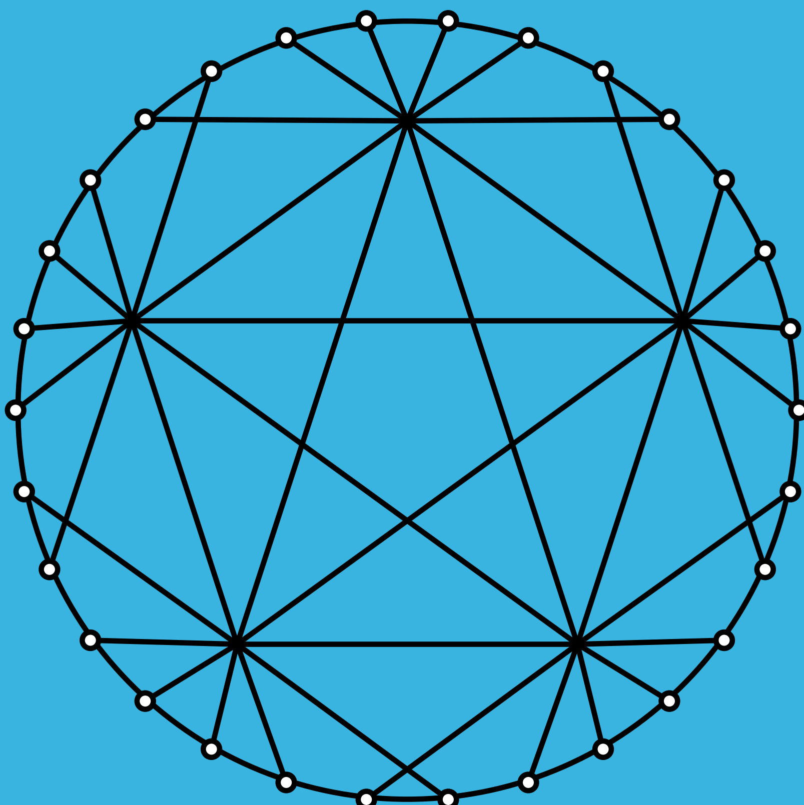

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(k, ℓ)-kernels in $P_m + e$

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Abstract

Let $k \geq 2$ and ℓ be positive integers. For a digraph D, a set $J \subseteq V(D)$ is said to be a (k, ℓ)-kernel of D if for all $x, y \in J$, $d_D(x, y) \geq k$ and for every $z \in V(D) \backslash J$, there exists $w \in J$ such that $d_D(z, w) \leq \ell$. In this paper, we give a necessary and sufficient condition for the existence of a (k, ℓ)-kernel in $P_m + e$, where P_m is a directed path on m vertices and e is any arc with both ends in P_m.

1 Introduction

For notation and terminology, in general, we follow [1]. Let D denote a finite digraph with vertex set $V(D)$ and arc set $A(D)$.

For $x, y \in V(D)$, *distance* from x to y in D, denoted by $d_D(x, y)$, is the number of arcs in a shortest directed path from x to y in D.

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For $X, Y \subseteq V(D)$, $d_D(X, Y) = \min\{d_D(x, y) : x \in X,\ y \in Y\}$. When $X = \{x\}$, denote $d_D(\{x\}, Y)$ by $d_D(x, Y)$. Similar notation holds for $Y = \{y\}$.

Let $k \geq 2$ and ℓ be positive integers. For a set $J \subseteq V(D)$, J is *k-independent* if for every $x, y \in J$, $d_D(x, y) \geq k$; J is *ℓ-absorbent* if for every $z \in V(D) \backslash J$, there exists $w \in J$ such that $d_D(z, w) \leq \ell$; J is a *(k, ℓ)-kernel* of D if J is both k-independent and ℓ-absorbent in D. A *k-kernel* is a $(k, k-1)$-kernel. From the definition of (k, ℓ)-kernel, it follows that, for $2 \leq k_0 \leq k$ and $\ell \leq \ell_0$, every (k, ℓ)-kernel of D is a (k_0, ℓ_0)-kernel of D.

The concept of kernel was introduced by von Neumann and Morgenstern (see [13]) as an abstract generalization of the concept of solution for cooperative games. Further, the study of kernels acquired its own significance due to its application in combinatorial games and Mathematical Logic. Generalizing the concept of kernels, Borowiecki and Kwaśnik (see [10]) introduced (k, ℓ)-kernels. A wide range of contributions are being made to the study of (k, ℓ)-kernels.

In [4], Galeana-Sánchez concentrated on the presence of (k, ℓ)-kernels in digraphs with symmetric pair of arcs. The behavior of (k, ℓ)-kernels in different products of digraphs were investigated by several authors like Kwaśnik, Szumny, Włoch and Włoch (see [10], [11], [12] and [14]). Their results included computing the number of (k, ℓ)-kernels in the respective product graphs. For additional results on (k, ℓ)-kernels, see [7]. Among the study on (k, ℓ)-kernels, a greater attention is to the study of k-kernels (see [5], [6] and [8]). In 1973, Chvátal (see [3]) had proved that the problem of determining whether a digraph admits a kernel or not is $\mathcal{NP}$-complete. Later, in 2014, it is proved in [6] that the problem of determining whether a digraph has a k-kernel, $k \geq 3$, is also $\mathcal{NP}$-complete.

Let P_m and C_m denote, respectively, a directed path and a directed cycle on m vertices. In [9], the authors provided a necessary and sufficient condition for P_m and C_m to have a (k, ℓ)-kernel.

Theorem 1.1. ([9]) *For $m \geq k$, P_m has a (k, ℓ)-kernel if, and only if, $k \leq \ell + 1$.*

Theorem 1.2. ([9]) *Let C_m be given with $m = nk + r$ and $n \geq 1$. Then C_m has a (k, ℓ)-kernel if, and only if, $k \leq \ell + 1$ and $r \leq n(\ell + 1 - k)$.*

In Section 2, we characterize the existence of (k, ℓ)-kernels in the digraph $P_m + e$, where e is any arc with both ends in P_m. In the way, Theorem 1.2 becomes a corollary.

2 (k,ℓ)-kernels in $P_m + e$

Let $P_m : x_1x_2x_3 \ldots x_m$ be a directed path on m vertices. Let $D = P_m + e$, where e is an arc with both ends in P_m. Given positive integers k and ℓ, where $2 \leq k \leq m$, in Theorems 2.1 and 2.5, we give a necessary and sufficient condition for the existence of a (k,ℓ)-kernel in $D = P_m + e$.

2.1 Unicyclic $P_m + e$

Let $D = P_m + x_bx_a$, where $1 \leq a < b \leq m$. Based on the conditions obtained, the statement gets split into six cases.

Theorem 2.1. *Let $D = P_m + x_bx_a$, where $1 \leq a < b \leq m$. For $2 \leq k \leq m$, let $a - 1 = t_1k + q_1$, $b - a + 1 = nk + r$ and $m - b = t_2k + q_2$, where $t_1, n, t_2 \geq 0$ and $0 \leq q_1, r, q_2 < k$.*

I. *If any one of the following conditions hold:*

   (i) $n \geq 1$ *and* $r = 0$;

   (ii) $n \geq 1$, $r \geq 1$, $q_2 \geq 1$ *and* $r + q_2 \leq k$;

   (iii) $n = t_1 = t_2 = 0$, $q_2 \geq 1$ *and* $r + q_2 \leq k < q_1 + r + q_2$;

   (iv) $n = t_1 = 0$ *and* $t_2 \geq 1$;

   (v) $n = 0$ *and* $t_1 \geq 1$;

   then D has a (k,ℓ)-kernel if, and only if, $k \leq \ell + 1$.

II. *Let $n \geq 1$, $r \geq 1$ and $q_2 = 0$. Then D has a (k,ℓ)-kernel if, and only if, $k \leq \ell + 1$ and $r \leq (n + t_2)(\ell + 1 - k)$.*

III. *Let $n \geq 1$ and $r + q_2 > k$. Then D has a (k,ℓ)- kernel if, and only if, $k \leq \ell + 1$ and $r + q_2 \leq (n + t_2)(\ell + 1 - k) + \ell + 1$.*

IV. *Let $n = t_1 = t_2 = 0$, $q_2 \geq 1$ and $q_1 + r + q_2 \leq k$. Then D has a (k,ℓ)-kernel if, and only if, $q_1 + r + q_2 \leq \ell + 1$.*

V. *Let $n = t_1 = t_2 = q_2 = 0$. Then D has a (k,ℓ)- kernel if, and only if,* $\max\{q_1 + 1, r\} \leq \ell + 1$.

VI. *Let $n = t_1 = t_2 = 0$ and $r + q_2 > k$. Then D has a (k,ℓ)- kernel if, and only if,* $\max\{q_1 + 1, r, q_2\} \leq \ell + 1$.

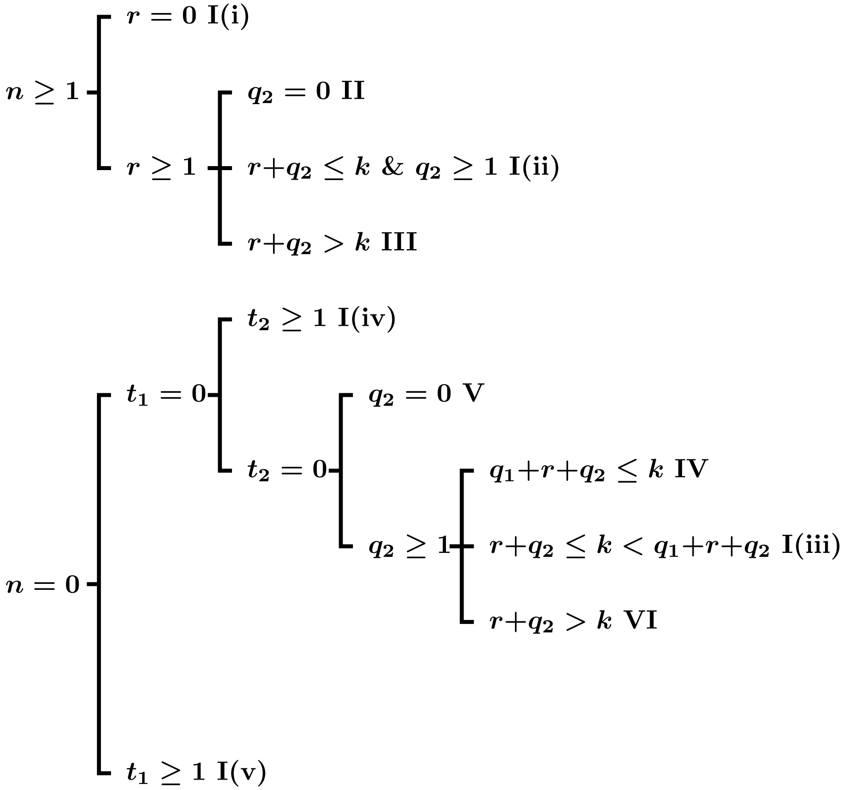

Figure 1.

Proof. Figure 1 shows that the cases are exhaustive. In the proof, the following facts are used often:

F1. If $k > \ell + 1$, then D has no (k, ℓ)-kernel J with $|J| \geq 2$.

Suppose that D has a (k, ℓ)-kernel J with $|J| \geq 2$. Then choose two vertices $x_i, x_j \in J$, $i < j$, with $x_c \notin J$ for $i < c < j$. Consider $d_D(x_{i+1}, J)$. If $d_D(x_{i+1}, J) = d_D(x_{i+1}, x_j)$, then $d_D(x_{i+1}, J) = d_D(x_i, x_j) - 1 \geq k - 1$. Otherwise, $d_D(x_{i+1}, J) = d_D(x_{i+1}, x_t)$ for some $t \neq j$; then $b \in \{i+1, i+2, \ldots, j-2\}$ and the shortest path from x_{i+1} to x_t is $x_{i+1}x_{i+2} \ldots x_b x_a x_{a+1} \ldots\ x_t$; and so $d_D(x_{i+1}, J) = d_D(x_i, x_t) - 1 \geq k - 1$. Hence, in both the possibilities, $d_D(x_{i+1}, J) \geq k - 1 > \ell$, a contradiction to the ℓ-absorbency of J.

F2. If $k > \ell + 1$ and D has a (k, ℓ)-kernel J, then $b = m$ and $J = \{x_i\}$, $i \in \{a, a+1, \ldots, m-1\}$.

By F1, $|J| = 1$. If $J = \{x_m\}$, then $d_D(x_1, x_m) = m - 1 \geq k - 1 > \ell$, a contradiction to the ℓ-absorbency of J. Hence $J = \{x_i\}$, $i \neq m$. If $i \in \{1, 2, \ldots, a-1\}$, then, no x_j, where $j \in \{a, a+1, \ldots, m\}$, is ℓ-absorbed by J. Therefore, $i \in \{a, a+1, \ldots, m-1\}$. If $b \neq m$, then $d_D(x_m, J) = \infty$, a contradiction. So $b = m$.

Proof of I(i). $n \geq 1$ and $r = 0$. (Then $b - a + 1 = nk$.)

First, assume $k \leq \ell + 1$. We consider two cases.

Case A. $q_2 = 0$. (Then $(t_1 + n + t_2)k + q_1 = m$.)

Let $J_0 = \{x_{k+q_1}, x_{2k+q_1}, x_{3k+q_1}, \ldots, x_{(t_1+n+t_2)k+q_1}\}$ and let

$$J = \begin{cases} J_0 & \text{if } q_1 = 0, \\ J_0 \cup \{x_{q_1}\} & \text{if } q_1 \geq 1. \end{cases}$$

As $k - 1 \leq \ell$, J is ℓ-absorbent. As $t_1 k + q_1 = a - 1$, $(t_1 + 1)k + q_1 = a - 1 + k$ and $(t_1 + n)k + q_1 = b$, we have: for k-independence, it is enough to check that $d_D(x_b, x_{a-1+k}) \geq k$. This inequality is true, since $d_D(x_b, x_{a-1+k}) = 1 + d_D(x_a, x_{a-1+k}) = 1 + k - 1 = k$.

Case B. $q_2 \geq 1$. (Then $(t_1 + n + t_2)k + q_1 + q_2 = m$.)

Let $J_0 = \{x_{q_1+q_2}, x_{k+q_1+q_2}, x_{2k+q_1+q_2}, \ldots, x_{(t_1+n+t_2)k+q_1+q_2}\}$ and let

$$J = \begin{cases} J_0 & \text{if } q_1 + q_2 \leq k, \\ J_0 \cup \{x_{q_1+q_2-k}\} & \text{if } q_1 + q_2 > k. \end{cases}$$

As $k - 1 \leq \ell$, J is ℓ-absorbent. We see that

$$\begin{aligned} (t_1 - 1)k + q_1 + q_2 &= a - 1 - k + q_2 \leq a - 2, \\ t_1 k + q_1 + q_2 &= a - 1 + q_2 \geq a, \\ (t_1 + n - 1)k + q_1 + q_2 &= b - k + q_2 \leq b - 1, \end{aligned}$$

and

$$(t_1 + n)k + q_1 + q_2 = b + q_2 \geq b + 1.$$

Therefore, for k-independence, it is enough to check that

$$d_D(x_{b-k+q_2}, x_{a-1+q_2}) \geq k.$$

This inequality holds true, because

$$\begin{aligned} d_D(x_{b-k+q_2}, x_{a-1+q_2}) &= d_D(x_{b-k+q_2}, x_b) + 1 + d_D(x_a, x_{a-1+q_2}) \\ &= k - q_2 + 1 + q_2 - 1 = k. \end{aligned}$$

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $k > \ell + 1$. Then, by F2, $b = m$ and $J = \{x_i\}$, $i \in \{a, a+1, \ldots, m-1\}$. Considering $x_{i+1} \in V(D) \backslash J$, we have $d_D(x_{i+1}, x_i) = nk - 1 \geq k - 1 > \ell$, a contradiction.

Proof of I(ii). $n \geq 1$, $r \geq 1$, $q_2 \geq 1$ and $r + q_2 \leq k$.

First, assume $k \leq \ell + 1$. Clearly, $q_1 + r + q_2 \leq 2k - 1$. Let

$$J_0 = \{x_{q_1+r+q_2}, x_{k+q_1+r+q_2}, x_{2k+q_1+r+q_2}, \ldots, x_{(t_1+n+t_2)k+q_1+r+q_2}\}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 + r + q_2 \leq k, \\ J_0 \cup \{x_{q_1+r+q_2-k}\} & \text{if } q_1 + r + q_2 > k. \end{cases}$$

Since $k - 1 \leq \ell$, J is ℓ-absorbent. As $(t_1 - 1)k + q_1 + r + q_2 = a - 1 - k + r + q_2 \leq a - 1$, $t_1 k + q_1 + r + q_2 = a - 1 + r + q_2 \geq a + 1$, $(t_1 + n - 1)k + q_1 + r + q_2 = b - k + q_2 \leq b - 1$ and $(t_1 + n)k + q_1 + r + q_2 = b + q_2 \geq b + 1$, we have: for k-independence, it is enough to check that $d_D(x_{b-k+q_2}, x_{a-1+r+q_2}) \geq k$. The inequality holds true, because $d_D(x_{b-k+q_2}, x_{a-1+r+q_2}) = d_D(x_{b-k+q_2}, x_b) + 1 + d_D(x_a, x_{a-1+r+q_2}) = k - q_2 + 1 + r + q_2 - 1 = k + r > k$.

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $k > \ell + 1$. Then, by F2, $b = m$. So $q_2 = 0$, a contradiction.

Proof of I(iii). $n = t_1 = t_2 = 0$, $q_2 \geq 1$ and $r + q_2 \leq k < q_1 + r + q_2$. (Then $q_1 \geq 1$ and $m = q_1 + r + q_2 < 2k$.)

First, assume $k \leq \ell + 1$. Let $J = \{x_{q_1+r+q_2-k}, x_{q_1+r+q_2}\}$. Since $k - 1 \leq \ell$, J is ℓ-absorbent. As $n = 0$, J is k-independent.

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $k > \ell + 1$. Then, by F2, $b = m$. So $q_2 = 0$, a contradiction.

Proof of I(iv). $n = t_1 = 0$ and $t_2 \geq 1$.

First, assume $k \leq \ell + 1$. Let

$$J_0 = \{x_{q_1+r+q_2}, x_{k+q_1+r+q_2}, x_{2k+q_1+r+q_2}, \ldots, x_{t_2k+q_1+r+q_2}\}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 + r + q_2 \leq k, \\ J_0 \cup \{x_{q_1+r+q_2-k}\} & \text{if } k < q_1 + r + q_2 \leq 2k, \\ J_0 \cup \{x_{q_1+r+q_2-2k}, x_{q_1+r+q_2-k}\} & \text{if } q_1 + r + q_2 > 2k. \end{cases}$$

As $n = 0$, J is k-independent and as $k-1 \leq \ell$, J is ℓ-absorbent.

Conversely, assume that D has a (k,ℓ)-kernel J. Suppose $k > \ell+1$. Then, by F2, $b = m$. So $t_2 = 0$, a contradiction.

Proof of I(v). $n = 0$ and $t_1 \geq 1$.

First, assume $k \leq \ell + 1$. Let

$$J_0 = \{x_{q_1+r+q_2}, x_{k+q_1+r+q_2}, x_{2k+q_1+r+q_2}, \ldots, x_{(t_1+t_2)k+q_1+r+q_2}\}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 + r + q_2 \leq k, \\ J_0 \cup \{x_{q_1+r+q_2-k}\} & \text{if } k < q_1 + r + q_2 \leq 2k, \\ J_0 \cup \{x_{q_1+r+q_2-2k}, x_{q_1+r+q_2-k}\} & \text{if } q_1 + r + q_2 > 2k. \end{cases}$$

As $n = 0$, J is k-independent and as $k-1 \leq \ell$, J is ℓ-absorbent.

Conversely, assume that D has a (k,ℓ)-kernel J. Suppose $k > \ell+1$. Then, by F2, $b = m$ and $J = \{x_i\}$, $i \in \{a, a+1, \ldots, m-1\}$. As $t_1 \geq 1$, $a \neq 1$. Considering $x_1 \in V(D)\backslash J$, $d_D(x_1, J) = d_D(x_1, x_i) = i - 1 \geq a - 1 = t_1k + q_1 \geq k > \ell + 1$, a contradiction.

Proof of II. $n \geq 1$, $r \geq 1$ and $q_2 = 0$.

First, assume $k \leq \ell+1$ and $r \leq (n+t_2)(\ell+1-k)$. So $(n+t_2)k + r \leq (n+t_2)(\ell+1)$. There exists a non-negative integer s such that $r = (n+t_2)s + w$, where $0 \leq w < n + t_2$. Clearly, $s \leq \ell + 1 - k$. We consider two cases.

Case A. $w = 0$. (Then $s \geq 1$.)

Let

$$J_0 = \{x_{k+q_1}, x_{2k+q_1}, x_{3k+q_1}, \ldots, x_{t_1k+q_1}, x_{t_1k+(k+s)+q_1}, \\ x_{t_1k+2(k+s)+q_1}, \ldots, x_{t_1k+(n+t_2)(k+s)+q_1}\}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 = 0, \\ J_0 \cup \{x_{q_1}\} & \text{if } q_1 \geq 1. \end{cases}$$

Clearly, $x_{a-1} \in J$ and $t_1k + (k+s) + q_1 \geq a + 2$. Also,

$$d_D(x_a, x_{t_1k+(k+s)+q_1}) = d_D(x_{t_1k+q_1+1}, x_{t_1k+(k+s)+q_1}) = k + s - 1 \geq k.$$

This implies if the shortest path from a vertex of J reaches $x_{t_1k+(k+s)+q_1}$ using the arc x_bx_a, then, the length of the shortest path is at least k. This, along with the definition of J, imply that J is k-independent. As $k + s - 1 \leq \ell$, J is ℓ-absorbent.

Case B. $w \geq 1$.

Clearly, $s \leq \ell - k$. (Otherwise, $s = \ell + 1 - k$, then $(n+t_2)(\ell+1-k) \geq r = (n+t_2)s + w \geq (n+t_2)\ (\ell+1-k) + 1$, a contradiction.) Let

$$\begin{aligned} J_0 = \{&x_{k+q_1}, x_{2k+q_1}, x_{3k+q_1}, \ldots, x_{t_1k+q_1}, x_{t_1k+(k+s)+q_1+1}, \\ &x_{t_1k+2(k+s)+q_1+2}, \ldots, x_{t_1k+w(k+s)+q_1+w}, x_{t_1k+(w+1)(k+s)+q_1+w}, \\ &\qquad x_{t_1k+(w+2)(k+s)+q_1+w}, \ldots, x_{t_1k+(n+t_2)(k+s)+q_1+w}\} \end{aligned}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 = 0, \\ J_0 \cup \{x_{q_1}\} & \text{if } q_1 \geq 1. \end{cases}$$

Clearly, $x_{a-1} \in J$ and $t_1k + (k+s) + q_1 + 1 = a + (k+s) \geq a + 2$. Also, $d_D(x_a, x_{t_1k+(k+s)+q_1+1}) = d_D(x_{t_1k+q_1+1}, x_{t_1k+(k+s)+q_1+1}) = k + s \geq k$. By a similar reason in Case A, J is k-independent. As $k + s \leq \ell$, J is ℓ-absorbent.

Conversely, assume that D has a (k,ℓ)-kernel J. We consider two cases.

Case a. $|J| = 1$.

Suppose $k > \ell + 1$. By F2, $b = m$ and $J = \{x_i\}$, where $i \in \{a, a+1, \ldots, m-1\}$. Therefore, $t_2 = 0$. Considering $x_{i+1} \in V(D)\backslash J$, we have $d_D(x_{i+1}, x_i) = nk + r - 1 \geq k - 1 > \ell$, a contradiction. Hence $k \leq \ell + 1$.

Suppose $r > (n+t_2)(\ell+1-k)$. Then $(n+t_2)k + r - 1 \geq (n+t_2)(\ell+1)$. If $J = \{x_m\}$, then $d_D(x_1, x_m) = m - 1 = (t_1 + n + t_2)k + q_1 + r - 1 \geq t_1k + (n+t_2)(\ell+1) + q_1 > \ell$, a contradiction. Thus $b = m$. If $J = \{x_i\}$, where $i \in \{1, 2, \ldots, a-1\}$, then $d_D(x_a, J) = \infty$, a contradiction. Therefore, $J = \{x_i\}$, $i \in \{a, a+1, \ldots, m-1\}$. But $d_D(x_{i+1}, x_i) = nk + r - 1 \geq nk + (n+t_2)(\ell+1-k) = n(\ell+1) + t_2(\ell+1-k) \geq \ell + 1$, a contradiction. Hence $r \leq (n+t_2)(\ell+1-k)$.

Case b. $|J| \geq 2$.

By F1, $k \leq \ell + 1$. Next, we show that $r \leq (n+t_2)(\ell+1-k)$.

Let i_1 be the least integer such that $a \leq i_1 \leq m$ and $x_{i_1} \in J$. Let $J_0 = J \cap \{x_a, x_{a+1}, \ldots, x_m\} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_q}\}$, where, one of the three possibilities occur: $b < i_1$ or $i_p \leq b < i_{p+1}$ for some p or $i_q \leq b$.

First, assume $b < i_1$. Now,

$$\begin{aligned} d_D(x_a, J) = d_D(x_a, x_{i_1}) &\geq 1 + d_D(x_a, x_b) \\ &= 1 + b - a = nk + r. \end{aligned}$$

If $r > (n+t_2)(\ell+1-k)$, then

$$\begin{aligned} d_D(x_a, J) &\geq nk + r \\ &> nk + (n+t_2)(\ell+1-k) \\ &= n(\ell+1) + t_2(\ell+1-k) > \ell, \end{aligned}$$

a contradiction. Hence $r \leq (n+t_2)(\ell+1-k)$.

Next, assume $i_p \leq b < i_{p+1}$. This implies $m > b$ and therefore, $t_2 \geq 1$. Also, $i_q = m$. If $d_D(x_{i_j}, x_{i_{j+1}}) \geq \ell+2$ for some $j \in \{1, 2, \dots, p-1, p+1, p+2, \dots, q-1\}$, then $d_D(x_{i_j+1}, J) = d_D(x_{i_j+1}, x_{i_{j+1}}) \geq \ell+1$, a contradiction. Hence, for every $j \in \{1, 2, \dots, p-1, p+1, p+2, \dots, q-1\}$, $d_D(x_{i_j}, x_{i_{j+1}}) \leq \ell+1$. Since $d_D(x_{i_j}, x_{i_{j+1}}) \geq k$, for $j = 1, 2, 3, \dots, p-1$, and $d_D(x_{i_p}, x_{i_1}) \geq k$, we have $pk \leq nk + r$ and so $p \leq n$. We consider two cases.

- $d_D(x_{i_p}, x_{i_1}) \leq \ell+1$: Then $nk + r \leq p(\ell+1) \leq n(\ell+1)$; i.e. $r \leq n(\ell+1-k)$. So $r \leq (n+t_2)(\ell+1-k)$.
- $d_D(x_{i_p}, x_{i_1}) \geq \ell+2$: Note that $d_D(x_a, x_{i_1}) \leq \ell$. If

  $$d_D(x_{i_p}, x_{i_{p+1}}) \geq \ell+2,$$

  then $d_D(x_{i_p+1}, J) \geq \ell+1$, a contradiction. Therefore,

  $$d_D(x_{i_p}, x_{i_{p+1}}) \leq \ell+1.$$

  Consequently, $(n+t_2)k + r - 1 = d_D(x_a, x_m) \leq (q-1)(\ell+1) + \ell$. This implies that $(n+t_2)k + r \leq q(\ell+1)$.

Observe that $q \leq n + t_2$. (Otherwise, $q > n + t_2$. As $p \leq n$, $q > n + t_2$ implies that $q - p - 1 \geq t_2$. Thus $d_D(x_b, x_m) = d_D(x_b, x_{i_{p+1}}) + d_D(x_{i_{p+1}}, x_{i_{p+2}}) + \cdots + d_D(x_{i_{q-1}}, x_m) \geq 1 + (q-p-1)k \geq 1 + t_2 k$, a contradiction to that $d_D(x_b, x_m) = m - b = t_2 k$.)

Hence $(n+t_2)k + r \leq q(\ell+1) \leq (n+t_2)(\ell+1)$ and therefore, $r \leq (n+t_2)(\ell+1-k)$.

Lastly, assume $i_q \leq b$. Then $b = m$. (Otherwise, by the definition of i_q, $d_D(x_m, J) = \infty$, a contradiction.) If $d_D(x_{i_j}, x_{i_{j+1}}) \geq \ell+2$ for some $j \in \{1, 2, \dots, q-1\}$, then $d_D(x_{i_j+1}, J) = d_D(x_{i_j+1}, x_{i_{j+1}}) \geq \ell+1$, a contradiction. Hence, for every $j \in \{1, 2, \dots, q-1\}$, $d_D(x_{i_j}, x_{i_{j+1}}) \leq \ell+1$. Now, if $d_D(x_{i_q}, x_{i_1}) \geq \ell+2$, then

$$d_D(x_{i_q+1}, J) = d_D(x_{i_q+1}, x_{i_1}) \geq \ell+1,$$

a contradiction. Therefore, $d_D(x_{i_q}, x_{i_1}) \leq \ell+1$. This implies $nk + r \leq q(\ell+1)$. Since $d_D(x_{i_j}, x_{i_{j+1}}) \geq k$, for $j = 1, 2, 3, \dots, q-1$, and

$d_D(x_{i_q}, x_{i_1}) \geq k$, we have $qk \leq nk + r$ and so $q \leq n$. Thus $nk + r \leq n(\ell+1)$; i.e. $r \leq n(\ell + 1 - k)$. So $r \leq (n + t_2)(\ell + 1 - k)$.

Proof of III. $n \geq 1$ and $r + q_2 > k$. (Then $r, q_2 \geq 2$.)

First, assume $k \leq \ell+1$ and $r+q_2-k \leq (n+t_2+1)(\ell+1-k)$. There exists a non-negative integer s such that $r + q_2 - k = (n + t_2 + 1)s + w$, where $0 \leq w < n + t_2 + 1$. Clearly, $s \leq \ell + 1 - k$. We consider two cases.

Case A. $w = 0$. (Then $s \geq 1$.)

Let

$$\begin{aligned} J_0 = \{&x_{k+q_1}, x_{2k+q_1}, x_{3k+q_1}, \ldots, x_{t_1k+q_1}, x_{t_1k+(k+s)+q_1}, \\ &x_{t_1k+2(k+s)+q_1}, \ldots, x_{t_1k+(n+t_2+1)(k+s)+q_1}\} \end{aligned}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 = 0, \\ J_0 \cup \{x_{q_1}\} & \text{if } q_1 \geq 1. \end{cases}$$

Clearly, $x_{a-1} \in J$ and $t_1k + (k+s) + q_1 \geq a + 2$. Also,

$$d_D(x_a, x_{t_1k+(k+s)+q_1}) = d_D(x_{t_1k+q_1+1}, x_{t_1k+(k+s)+q_1}) = k + s - 1 \geq k.$$

By a similar reason in Case A of II, J is k-independent and ℓ-absorbent.

Case B. $w \geq 1$.

Clearly, $s \leq \ell - k$. (Otherwise, $s = \ell + 1 - k$, then $(n + t_2 + 1)(\ell + 1 - k) \geq r + q_2 - k = (n+t_2+1)s + w \geq (n+t_2+1)(\ell+1-k)+1$, a contradiction.)

Let

$$\begin{aligned} J_0 = \{&x_{k+q_1}, x_{2k+q_1}, x_{3k+q_1}, \ldots, x_{t_1k+q_1}, \\ &x_{t_1k+k+s+q_1+1}, x_{t_1k+2(k+s)+q_1+2}, \\ &\ldots, x_{t_1k+w(k+s)+q_1+w}, x_{t_1k+(w+1)(k+s)+q_1+w}, \\ &x_{t_1k+(w+2)(k+s)+q_1+w}, \ldots, x_{t_1k+(n+t_2+1)(k+s)+q_1+w}\} \end{aligned}$$

and let

$$J = \begin{cases} J_0 & \text{if } q_1 = 0, \\ J_0 \cup \{x_{q_1}\} & \text{if } q_1 \geq 1. \end{cases}$$

Clearly, $x_{a-1} \in J$ and $t_1k + (k+s) + q_1 + 1 = a + (k+s) \geq a + 2$. Also, $d_D(x_a, x_{t_1k+(k+s)+q_1+1}) = d_D(x_a, x_{a+(k+s)}) = k + s \geq k$. By a similar reason in Case B of II, J is k-independent and ℓ-absorbent.

Conversely, assume that D has a (k,ℓ)-kernel J. We consider two cases.

Case a. $|J| = 1$.

Suppose $J = \{x_i\}$, $i \neq m$. As $q_2 \geq 2$, $b \neq m$ and therefore, $d_D(x_m, J) = \infty$, a contradiction. Thus $J = \{x_m\}$. By F2, $k \leq \ell+1$. Also, $r+q_2-k \leq (n+t_2+1)(\ell+1-k)$; otherwise, $r+q_2 > (n+t_2+1)(\ell+1-k)+k$, then $d_D(x_1, x_m) = m-1 = (t_1+n+t_2)k+q_1+r+q_2-1 \geq (t_1+n+t_2)k+q_1+(n+t_2+1)(\ell+1-k)+k = t_1k+(n+t_2+1)(\ell+1)+q_1 \geq 2(\ell+1) > \ell$, a contradiction.

Case b. $|J| \geq 2$.

By F1, $k \leq \ell+1$. Next, we show that $r+q_2-k \leq (n+t_2+1)(\ell+1-k)$; i.e. $r+q_2 \leq (n+t_2)(\ell+1-k)+\ell+1$.

Let i_1 be the least integer such that $a \leq i_1 \leq m$ and $x_{i_1} \in J$. Let $J_0 = J \cap \{x_a, x_{a+1}, \ldots, x_m\} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_q}\}$. Clearly, $i_q = m$. Note that either $b < i_1$ or $i_p \leq b < i_{p+1}$ for some p.

First, assume $b < i_1$. Now,

$$d_D(x_a, J) = d_D(x_a, x_{i_1}) \geq 1 + d_D(x_a, x_b) = b - a + 1 = nk + r.$$

If $r > (n+t_2)(\ell+1-k)+\ell+1-q_2$, then

$$\begin{aligned} d_D(x_a, J) &> nk + (n+t_2+1)(\ell+1-k) + k - q_2 \\ &\geq n(\ell+1) + (t_2+1)(\ell+1-k) + 1 \\ &\geq \ell+2, \end{aligned}$$

a contradiction. Therefore, $r+q_2 \leq (n+t_2)(\ell+1-k)+\ell+1$.

Next, assume $i_p \leq b < i_{p+1}$. If $d_D(x_{i_j}, x_{i_{j+1}}) \geq \ell+2$ for some $j \in \{1, 2, \ldots, p-1, p+1, p+2, \ldots, q-1\}$, then

$$d_D(x_{i_j+1}, J) = d_D(x_{i_j+1}, x_{i_{j+1}}) \geq \ell+1,$$

a contradiction. Hence, for every $j \in \{1, 2, \ldots, p-1, p+1, p+2, \ldots, q-1\}$, $d_D(x_{i_j}, x_{i_{j+1}}) \leq \ell+1$. Since $d_D(x_{i_j}, x_{i_{j+1}}) \geq k$, for $j = 1, 2, 3, \ldots, p-1$, and $d_D(x_{i_p}, x_{i_1}) \geq k$, we have, $pk \leq nk+r$ and so $p \leq n$. We consider two cases.

- $d_D(x_{i_p}, x_{i_1}) \leq \ell+1$: Then $nk+r \leq p(\ell+1)$. Thus $r \leq p(\ell+1-n) \leq n(\ell+1-k) \leq (n+t_2)(\ell+1-k)$. We know that $q_2 \leq k-1 \leq \ell < \ell+1$. Therefore, combining both the inequalities, we have $r+q_2 < (n+t_2)(\ell+1-k)+\ell+1$.

- $d_D(x_{i_p}, x_{i_1}) \geq \ell+2$: If $d_D(x_{i_p}, x_{i_{p+1}}) \geq \ell+2$, then, $d_D(x_{i_p+1}, J) \geq \ell+1$, a contradiction. Thus $d_D(x_{i_p}, x_{i_{p+1}}) \leq \ell+1$. Note that $d_D(x_a, x_{i_1}) \leq \ell$. Now, $(n+t_2)k+r+q_2-1 = m-a = d_D(x_a, x_m) \leq (q-1)(\ell+1)+\ell$ and thus $(n+t_2)k+r+q_2 \leq q(\ell+1)$.

Observe that $q \leq n + t_2 + 1$. (Otherwise, $q > n + t_2 + 1$. As $p \leq n$, $q - p - 1 \geq q - n - 1 \geq t_2 + 1$. Thus $d_D(x_b, x_m) = d_D(x_b, x_{i_{p+1}}) + d_D(x_{i_{p+1}}, x_{i_{p+2}}) + \cdots + d_D(x_{i_{q-1}}, x_m) \geq 1 + (q-p-1)k \geq 1 + (t_2+1)k$, a contradiction to that $d_D(x_b, x_m) = m - b = t_2 k + q_2 < (t_2 + 1)k$.)

Hence $(n + t_2)k + r + q_2 \leq q(\ell + 1) \leq (n + t_2 + 1)(\ell + 1)$ and therefore, $r + q_2 \leq (n + t_2)(\ell + 1 - k) + \ell + 1$.

Proof of IV. $n = t_1 = t_2 = 0$, $q_2 \geq 1$ and $q_1 + r + q_2 \leq k$.

Then $m = q_1 + r + q_2$ and so $m = k$.

First, assume $m \leq \ell + 1$. Let $J = \{x_m\}$. Clearly, J is k-independent. As $d_D(x_1, x_m) = m - 1 \leq \ell$, J is ℓ-absorbent.

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $m > \ell + 1$. As $m = k$, $|J| = 1$. Clearly, $J = \{x_m\}$. But $d_D(x_1, x_m) = m - 1 > \ell$, a contradiction.

Proof of V. $n = t_1 = t_2 = q_2 = 0$. (Then $m = b = q_1 + r \leq 2k - 2$.)

First, assume $\max\{q_1 + 1, r\} \leq \ell + 1$. Let $J = \{x_a\}$. As $|J| = 1$, J is k-independent. If $q_1 + 1 \leq r$, then $r \leq \ell + 1$; as $d_D(x_{a+1}, J) = d_D(x_{a+1}, x_a) = r - 1 \leq \ell$ and $d_D(x_1, J) = d_D(x_1, x_a) = a - 1 = q_1 \leq r - 1 \leq \ell$, J is ℓ-absorbent. If $q_1 + 1 > r$, then $q_1 \leq \ell$; as $d_D(x_{a+1}, J) = d_D(x_{a+1}, x_a) = r - 1 < q_1 \leq \ell$ and $d_D(x_1, J) = d_D(x_1, x_a) = a - 1 = q_1 \leq \ell$, J is ℓ-absorbent.

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $\max\{q_1 + 1, r\} \geq \ell + 2$. As $k \geq \max\{q_1 + 1, r\} \geq \ell + 2$, by $F2$, $b = m$ and $J = \{x_i\}$, $i \in \{a, a + 1, \ldots, m - 1\}$. If $\max\{q_1 + 1, r\} = r$, then $d_D(x_{i+1}, J) = d_D(x_{i+1}, x_i) = r - 1 \geq \ell + 1$, a contradiction. If $\max\{q_1 + 1, r\} = q_1 + 1$, then $d_D(x_1, J) = d_D(x_1, x_i) = i - 1 \geq a - 1 = q_1 \geq \ell + 1$, a contradiction.

Proof of VI. $n = t_1 = t_2 = 0$ and $r + q_2 > k$. (Then $r, q_2 \geq 2$.)

First, assume $\max\{q_1 + 1, r, q_2\} \leq \ell + 1$. We consider two cases.

Case A. $q_1 = k - 1$.

Then $\max\{q_1 + 1, r, q_2\} = q_1 + 1 = k$. Let

$$J = \{x_{q_1+r+q_2-2k}, x_{q_1+r+q_2-k}, x_{q_1+r+q_2}\}.$$

As $n = 0$, J is k-independent and as $k \leq \ell + 1$, J is ℓ-absorbent.

Case B. $q_1 \leq k - 2$.

Let $J = \{x_a, x_m\}$. As $d_D(x_a, x_m) = m - a = r + q_2 - 1 \geq k + 1$, J is k-independent. We see that $d_D(x_1, x_a) = a - 1 = q_1 \leq \max\{q_1 +$

$1, r, q_2\} - 1 \leq \ell$, $d_D(x_{a+1}, x_a) = r - 1 \leq \max\{q_1 + 1, r, q_2\} - 1 \leq \ell$ and $d_D(x_{b+1}, x_m) = m - b - 1 = q_2 - 1 \leq \max\{q_1 + 1, r, q_2\} - 1 \leq \ell$. Thus J is ℓ-absorbent.

Conversely, assume that D has a (k, ℓ)-kernel J. Suppose $\max\{q_1 + 1, r, q_2\} > \ell + 1$. Then $k > \ell + 1$. By F2, $b = m$. So $q_2 = 0$, a contradiction.

This completes the proof. □

Theorem 1.2 is a corollary to Theorem 2.1.

Corollary 2.2. *Let $D = P_m + x_b x_a$, where $1 \leq a < b \leq m$. For $2 \leq k \leq m$, let $a-1 = t_1 k + q_1$, $b-a+1 = nk+r$ and $m-b = t_2 k + q_2$, where $t_1, n, t_2 \geq 0$ and $0 \leq q_1, r, q_2 < k$. Then, D has a k-kernel if, and only if, none of the following conditions hold:*

- *$n \geq 1$, $r \geq 1$ and $q_2 = 0$;*
- *$n \geq 1$ and $r + q_2 > k$.*

Proof. Assume that there exists a k-kernel in either of the cases. Putting $\ell = k-1$ in II and III of Theorem 2.1, we have $r \leq (n+t_2)(k-1+1+k) = 0$ and $r + q_2 \leq (n + t_2 + 1)(k - 1 + 1 + k) + k - 1 + 1 = k$, a contradiction to the respective hypothesis.

Converse part directly holds from the proof of Theorem 2.1. □

The above corollary adds an additional family to the existing classes of digraphs with k-kernels.

2.2 Acyclic $P_m + e$

Let $D = P_m + x_a x_b$, where $1 \leq a < b \leq m$. Here, D is a digraph with no directed cycle. A well-known theorem of Richardson on the existence of kernel is Theorem 2.3.

Theorem 2.3. *Every digraph without directed cycles has a* $(2, 1)$*-kernel.*

In [2], Bród, Włoch and Włoch have proved the following result on k-kernels as a generalized form of Theorem 2.3.

Theorem 2.4. ([2]) *A digraph without directed cycles has a* k*-kernel, for* $k \geq 2$.

We make use of the above theorem for the "if" part of our result.

Theorem 2.5. *Let* $D = P_m + x_a x_b$*, where* $1 \leq a < b \leq m$ *and* $a + 1 \neq b$*. For* $2 \leq k \leq m$*,* D *has a* (k, ℓ)*-kernel if, and only if,* $k \leq \ell + 1$.

Proof. By Theorem 2.4, D has a k-kernel. Therefore, for every $\ell \geq k - 1$, D has a (k, ℓ)-kernel.

Conversely, assume that D has a (k, ℓ)-kernel J. Then $x_m \in J$. Suppose $k > \ell+1$. If $J = \{x_m\}$, then $d_D(x_1, x_m) = m-1 \geq k-1 > \ell$, a contradiction. Thus $|J| \geq 2$. Choose two vertices $x_i, x_j \in J$, $i < j$, with $x_c \notin J$ for $i < c < j$. If $d_D(x_{i+1}, J) = d_D(x_{i+1}, x_j)$, then $d_D(x_{i+1}, J) = d_D(x_i, x_j) - 1 \geq k - 1$. Otherwise, $d_D(x_{i+1}, J) = d_D(x_{i+1}, x_t)$ for some $t \neq j$; then $i+1 \leq a < b \leq t$ and the shortest path from x_{i+1} to x_t is $x_{i+1}x_{i+2} \ldots x_a x_b x_{b+1} \ldots x_t$; thus $d_D(x_{i+1}, J) = d_D(x_i, x_t) - 1 \geq k - 1$. Hence, in both the possibilities, $d_D(x_{i+1}, J) \geq k - 1 > \ell$, a contradiction.

This completes the proof. □

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