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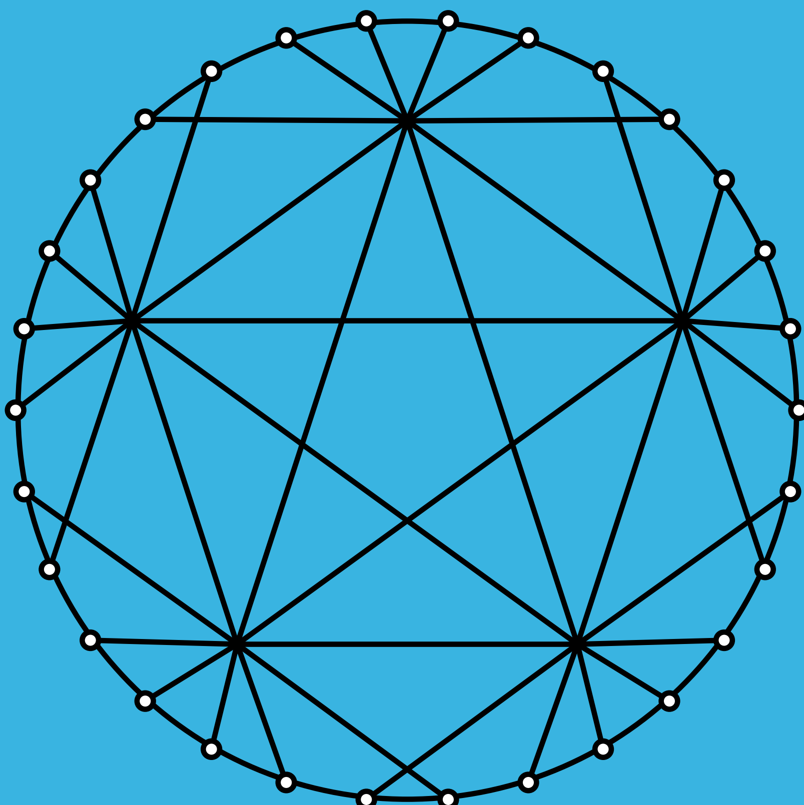

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Decomposition of complete graphs into connected unicyclic bipartite graphs with seven edges

Dalibor Froncek*[1] and Michael Kubesa[2]

[1]University of Minnesota Duluth, MN, USA
dalibor@d.umn.edu
[2]VŠB-Technical University of Ostrava, Czech Republic
michael.kubesa@vsb.cz

Abstract

In this paper, we completely determine the spectrum of connected unicyclic bipartite graphs with seven edges decomposing complete graphs.

1 Introduction

Graph decompositions is a classical graph theory topic that has been extensively studied for decades. In particular, decompositions of complete graphs into mutually isomorphic subgraphs has attracted most attention.

*Corresponding author.
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A common approach is to define a class of graphs, often infinite, and determine which complete graphs admit a decomposition into such graphs. Probably the best known example of this is the Ringel Conjecture [24] that every tree on $n+1$ vertices decomposes the complete graph K_{2n+1}.

Another approach is to classify all graphs with a given number of vertices and/or edges and determine which complete graphs they decompose. It seems that all connected graphs with up to six edges have been fully classified, as well as some classes of graphs with seven edges, and almost all graphs with eight edges. An overview of related results is presented in Section 3.

We continue in this direction by classifying all connected unicyclic bipartite graphs with seven edges decomposing complete graphs.

Our methods are mostly based on Rosa-type labelings, introduced by Rosa in 1967 [25].

2 Definitions and tools

We start with some basic definitions. A *unicyclic graph* is a simple finite graph without loops containing exactly one cycle.

Definition 2.1. Let H be a graph. A *decomposition* of the graph H is a collection of pairwise edge-disjoint subgraphs $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ such that every edge of H appears in exactly one subgraph $G_i \in \mathcal{D}$.

We say that the collection forms a *G-decomposition* of H (also known as an *(H, G)-design*) if each subgraph G_r is isomorphic to a given graph G. If H is the complete graph K_n, then we can use just the term *G-design*.

Because we focus solely on decompositions of complete graphs, we only use the term G-decomposition or G-design.

Definition 2.2. A G-decomposition of the complete graph K_n is *cyclic* if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, n-1$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo n.

Definition 2.3. A G-decomposition of K_n is 1-*rotational* if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation φ

of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, n-2$ and $\varphi(x_{n-1}) = x_{n-1}$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo $n-1$.

We will use the interval notation $[k, n]$ for the set of consecutive integers $\{k, k+1, k+2, \ldots, n\}$.

One of the basic and most useful tools for finding G-designs is the following labeling.

Definition 2.4 (Rosa [25])**.** Let G be a graph with n edges. A *ρ-labeling* (sometimes also called *rosy labeling*) of G is an injective function

$$f\colon V(G) \to [0, 2n]$$

that induces the *length function* $\ell\colon E(G) \to [1, n]$ defined as

$$\ell(uv) = \min\left\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\right\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n].$$

A graph G possessing a ρ-labeling decomposes the complete graph, as proved by Rosa in 1967.

Theorem 2.5 (Rosa [25])**.** *Let G be a graph with n edges. A cyclic G-decomposition of the complete graph K_{2n+1} exists if and only if G admits a ρ-labeling.*

When a graph G with n edges has a vertex w of degree one and $G - w$ admits a ρ-labeling, a modification of ρ-labeling can be used to find a G-decomposition of K_{2n}. Such labeling is known as 1-*rotational ρ-labeling* and was first used by Huang and Rosa in [17], although a formal definition was not stated there.

Definition 2.6 (Huang, Rosa [17])**.** Let G be a graph with n edges and edge ww' where $\deg(w) = 1$. A 1-*rotational ρ-labeling* of G consists of an injective function $f\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ with $f(w) = \infty$ that induces a *length function* $\ell\colon E(G) \to [1, n-1] \cup \{\infty\}$ which is defined as

$$\ell(uv) = \min\left\{|f(u) - f(v)|, 2n - 1 - |f(u) - f(v)|\right\}$$

for $u, v \neq w$ and

$$\ell(ww') = \infty$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n-1] \cup \{\infty\}.$$

This technique was used in [17] and proved only for particular graphs studied in that paper. The following theorem is considered folklore.

Theorem 2.7. *Let G be a graph with n edges. If G admits a 1-rotational ρ-labeling, then there exists a 1-rotational G-decomposition of the complete graph K_{2n}.*

One can observe that a necessary condition for K_n to admit a G-design for a graph G with 7 edges is that the number of edges in K_n must be divisible by 7, implying $n \equiv 0, 1 \pmod 7$. For the graphs we are interested in, the above theorems only allow decompositions of K_{14} and K_{15}. Therefore, we will need additional tools, which are some more restrictive modifications of ρ-labeling.

Definition 2.8 (Rosa [25])**.** Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. An *α-labeling* of G is a ρ-labeling f with the additional property that there exist λ such that $f(u) \leq \lambda < f(v) \leq n$ for every $u \in U$ and $v \in V$. The length function is then defined as

$$\ell(uv) = f(v) - f(u).$$

There are also labelings that are less restrictive yet also produce G-decompositions of larger complete graphs; that is, K_{2nk+1} for any $k \geq 1$ when G has n edges.

Definition 2.9 (El-Zanati, Vanden Eynden [8])**.** Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. A *σ^+-labeling* of G is a ρ-labeling f with the additional property that for every $u \in U$ and $v \in V$ if $uv \in E(G)$, then $f(u) < f(v)$ and the length function is defined as

$$\ell(uv) = f(v) - f(u).$$

The σ^+-labeling is a generalization of the α-labeling and can be viewed as "locally α-labeling." Not all labels in set U need to be smaller that all labels in V, but rather only labels of all neighbors of a given vertex $u \in U$ have to be larger than that of u and vice versa, all neighbors of $v \in V$ have to have labels smaller than the label of v. Even the relaxed conditions guarantee decompositions of K_{2nk+1}, as proved by El-Zanati and Vanden Eynden [8].

Theorem 2.10 (El-Zanati, Vanden Eynden [8]). *Let G be a bipartite graph with n edges. If G admits a σ^+-labeling, then there exists a cyclic G-decomposition of the complete graph K_{2nk+1} for every $k \geq 1$.*

To decompose complete graphs with $2nk$ vertices into graphs with n edges, we will use the 1-rotational σ^+-labeling. Although the technique using such labeling has been used before (see, e.g., [9]), a formal definition has not been introduced yet.

Definition 2.11. Let G be a bipartite graph with n edges, vertex w of degree one and an edge ww'. A *1-rotational σ^+-labeling* of G is a 1-rotational ρ-labeling with the additional property that for every $u \in U$ and $v \in V$ if $u, v \neq w$ and $uv \in E(G)$, then $f(u) < f(v)$ and the length function is defined as

$$\ell(uv) = f(v) - f(u)$$

for $u, v \neq w$ and

$$\ell(ww') = \infty.$$

It is easy to see that when we have a σ^+-labeling where the longest edge is uw, vertex w is of degree one and all other vertices have labels at most $2n - 2$, the labeling can be transformed to a 1-rotational σ^+-labeling.

Observation 2.12. *Let G be a bipartite graph with n edges, an edge uw where w is of degree one and a σ^+-labeling f. If $f(w) > f(x)$ for every $x \neq w$ and $\ell(uw) = n$, then there exists a 1-rotational σ^+-labeling $g\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ defined as $g(x) = f(x)$ for $x \neq w$ and $g(w) = \infty$.*

The following analogue of the above theorems was proved recently.

Theorem 2.13 (Fahnenstiel, Froncek [9]). *Let G be a bipartite graph with n edges and a vertex of degree one. If G admits a 1-rotational σ^+-labeling, then there exists a 1-rotational G-decomposition of the complete graph K_{2nk} for every $k \geq 1$.*

In our constructions, we will also need to decompose complete bipartite graphs. The tools are similar, based on labelings as well. An equivalent of ρ-labeling for bipartite graphs is called bilabeling and has been used for years by numerous authors; it is considered folklore.

Definition 2.14. Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. An *α-bilabeling* of G is a function $f : V(G) \to [0, n-1]$

that is injective when restricted to sets U and V, respectively, with the additional properties that there exist λ such that $f(u) \leq \lambda < n$ and for every $u \in U$ and $v \in V$ and there is also an induced length function defined as

$$\ell(uv) = f(v) - f(u)$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [0, n-1].$$

The following theorem is also considered folklore.

Theorem 2.15. *Let G be a bipartite graph with n edges. If G admits an α-bilabeling, then there exists a G-decomposition of the complete bipartite graph $K_{nk,nm}$ for every $k, m \geq 1$.*

It is easy to observe that every α-labeling f can be transformed into an α-bilabeling f' by setting $f'(u) = f(u)$ for every $u \in U$ and $f'(v) = f(v) - 1$ for every $v \in V$. Thus, we have the following corollary.

Corollary 2.16. *Let G be a bipartite graph with n edges. If G admits an α-labeling, then there exists a G-decomposition of the complete bipartite graph $K_{nk,nm}$ for every $k, m \geq 1$.*

3 Related results

It seems that there has been no attempt to completely determine the decomposition spectrum for graphs with seven edges. We first summarize what is known about classification of graphs of similar size, that is, graphs where $|E(G)| \leq 8$.

3.1 Graphs with at most four edges

The cases of one or two edges are trivial. For three edges, the graph is either a K_3, $K_{1,3}$, P_4, $P_2 \cup P_3$ or $3K_2$. The case of K_3 is a Steiner triple system, whose existence was solved by Kirkman [20]. The paw (or claw) $K_{1,3}$ was settled by Cain [5] and the path P_4 by Bermond [1]. The union of paths $P_2 \cup P_3$ was solved by Bermond, Huang, Rosa and Sotteau [2].

The matching $M_3 = 3K_2$ was solved along with all other matchings by de Werra [27].

There are five connected graphs with four edges. The case of C_4 was classified by Kotzig [21] and the unicyclic graph with a triangle by Bermond and Schönheim [3]. The path P_5 and the tree with a unique vertex of degree three were settled by Huang and Rosa [17], and the star $K_{1,4}$ by Yamamoto et al. [28].

The forests with six vertices, that is, $2P_3, P_4 \cup P_2$ and $K_{1,3} \cup K_2$ were classified by Yin and Gong [29].

We believe that the case of $P_3 \cup 2P_2$ is a folklore, and the results follow for instance from the existence of its σ^+- or 1-rotational σ^+-labeling. Finally, matching $M_4 = 4K_2$ was solved by de Werra [27] as already mentioned above.

3.2 Graphs with five edges

Bermond, Huang, Rosa and Sotteau [2] studied the decomposition spectrum for all graphs with at most five vertices. They determined it completely for graphs with at most five edges.

Huang and Rosa [17] classified the spectra for all trees on up to nine vertices.

Graphs with five edges and more than five vertices that are not trees must be disconnected. They are either forests, or $C_4 \cup K_2$, or contain one triangle. Those containing a cycle were settled by Yin and Gong [29] except for $K_3 \cup 2K_2$. We were unable to find any results on forests with five edges except when it is a matching, which was solved by de Werra [27].

3.3 Graphs with six edges

The only graph on four vertices with six edges is K_4 and was treated by Hanani [16]. The graphs on five vertices have been completely settled by Bermond, Huang, Rosa and Sotteau [2] except for a decomposition of K_{24} into $\overline{P_5}$, the complement of P_5. This case was solved by Blinco [4] and independently by Kang and Wang [19].

Yin and Gong [29] found necessary and sufficient conditions for the existence of a G-design for graphs with six vertices and $3 \leq |E(G)| \leq 6$.

Graphs with six edges and more than six vertices are either trees (treated in [17]) or disconnected. We were unable to find any results on the disconnected graphs.

3.4 Graphs with seven edges

We focus on graphs with seven edges. Those with five vertices were investigated in [2], [4], and [19]. Cui [7], Blinco [4], and Tian, Du, and Kang [26] studied connected graphs with seven edges and six vertices.

Theorem 3.1 (Cui [7], Blinco [4], Tian et al. [26]). *There exists a G-decomposition of K_n into a connected graph G on six vertices and seven edges if and only if the necessary conditions are met except for eight exceptions when $n = 7$ or $n = 8$.*

The only disconnected graph with seven edges and six vertices is $K_4 \cup K_2$. The spectrum for this graph was determined by Tian, Du and Kang [26].

All graphs with seven edges and seven vertices are either unicyclic or disconnected, and we investigate the connected bipartite ones in the following section.

Connected graphs with seven edges and eight vertices are trees, which were investigated by Huang and Rosa [17]. We are not aware of any attempt to classify graphs (necessarily disconnected) with seven edges and more than eight vertices, or disconnected graphs with seven edges and seven vertices.

3.5 Graphs with eight edges

Graphs with eight edges have been extensively studied recently (see, e.g., [6, 9, 10, 11, 12, 13, 14, 15, 18]) and there are fewer than ten graphs, all of them disconnected, where the spectrum is not known.

4 Non-existence results

In this section, we present proofs of non-existence of G-decompositions of K_n for certain graphs G and $n = 7, 8$.

We denote the graphs decomposing K_n by G_i, where $i = 1, 2, 3$ when $n = 7$ and $i = 1, 2, 3, 4$ when $n = 8$. By $\deg_{G_i}(x_j)$ we denote the degree of vertex x_j in G_i. The *graph degree sequence* of a graph G_i is the list of the vertex degrees in G_i and will be denoted by $\text{gds}(G_i) = (a_1, a_2, \ldots, a_n)$, where n is the order of the respective graph K_n and $a_j = \deg_{G_i}(x_j)$. It will be listed in such a way that the sequence for G_i will be non-increasing. Notice that for $n = 8$ we will have the last entry in $\text{gds}(G_i)$ equal to zero. To simplify our arguments when $n = 8$, we will consider the vertex of degree zero to belong to the respective copy G_i.

The *vertex degree sequence* of a vertex x_j is the list of the degrees of x_j in the three or four copies of G and will be denoted by $\text{vds}(x_j)$. By *vertex degree set* we understand the unordered multiset of degrees of a particular vertex, denote it $\text{DS}(x_j)$ and usually list in non-increasing order.

For convenience, we first present a catalog of all connected unicyclic bipartite graphs with seven edges in Figure 1. We use the notation introduced by Reed and Wilson in [23].

Proposition 4.1. *The graphs $B39, B40$ and $B46$ do not decompose K_7.*

Proof. We proceed by contradiction and assume such decompositions exist.

First we observe that the three graphs are factors of K_7 and thus the vertex degree sets related to a potential decomposition cannot contain any 0's.

For graph $B39$, let $\deg_{G_1}(x_1) = 4$ and $\deg_{G_1}(x_2) = 3$. Hence, the edge x_1x_2 is not in G_1. Because we have $\deg_{G_1}(x_1) = 4$, it follows that $\deg_{G_2}(x_1) = 1$ and also $\deg_{G_3}(x_1) = 1$. We can now assume without loss of generality that x_1x_2 is an edge in G_2. But then we must have $\deg_{G_2}(x_2) \geq 3$, and the sum of degrees of x_2 is already equal to six. This leaves no room for edges incident with x_2 in G_3, and $B39$ cannot decompose K_7.

For graph $B40$, let $\deg_{G_1}(x_1) = 5$. Because $B40$ is a factor of K_7, we must have $\deg_{G_i}(x_1) \geq 1$ for $i = 2, 3$. This is impossible, as then x_1 would be of degree at least seven in K_7.

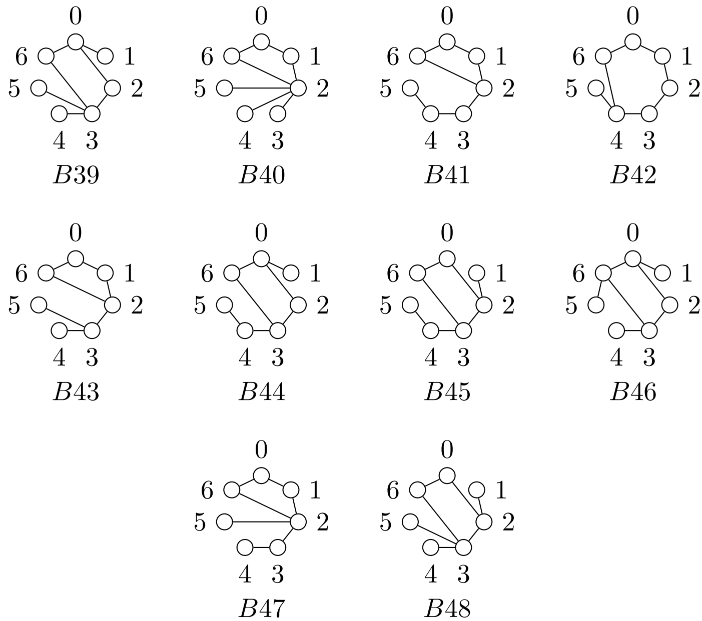

Figure 1: The connected unicyclic bipartite graphs with 7 edges $B39$–$B48$

The graph degree sequence of $B46$ is $\mathrm{gds}(B46) = (3,3,3,2,1,1,1)$. Then $\deg_{G_1}(x_j) = 3$ for $j = 1,2,3$ and without loss of generality we can assume the vertex degree sequence $\mathrm{vds}(x_1) = (3,2,1)$, since the degrees in K_7 must add up to six. Similarly, for x_2 we must have $\mathrm{vds}(x_2) = (3,2,1)$ or $(3,1,2)$. But the former is impossible, since it would mean that in copy G_2 there are two vertices of degree two, which is not the case. For the same reasons, we must have $\mathrm{vds}(x_3) = (3,2,1)$ or $(3,1,2)$, but none of them is possible, as either G_2 or G_3 would then have two vertices of degree two. This completes the proof. □

Proposition 4.2. *The graphs $B39$ and $B40$ do not decompose K_8.*

Proof. We again proceed by contradiction, assuming such decompositions exist.

For graph $B39$ we observe that every edge has one endvertex of degree three or four, and the other of degree one or two. Let x_1, x_2, x_3, x_4 be the vertices of the four-cycle in G_1 such that $\deg_{G_1}(x_1) = 4$ and $\deg_{G_1}(x_3) = 3$. By our observation above, assuming the edge x_1x_3 is in G_2, we have one of x_1, x_3 of degree at least three in G_2, because every edge in $B39$ is incident with a vertex of degree either 4 or 3.

Therefore we have a vertex x_j, $j \in \{1,3\}$ such that $\deg_{G_1}(x_j) \geq 3$ and $\deg_{G_2}(x_j) \geq 3$.

Now we consider four-cycle $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}$ in G_3. Let $\deg_{G_3}(x_{i_1}) = 4$ and $\deg_{G_3}(x_{i_3}) = 3$. The edge $x_{i_1}x_{i_3}$ belongs to G_s, $s \in \{1,2,4\}$.

We again obtain that there exists a vertex x_{i_k} with $k \in \{1,3\}$ such that $\deg_{G_3}(x_{i_k}) \geq 3$ and $\deg_{G_s}(x_{i_k}) \geq 3$. Obviously, $j \neq i_k$, otherwise

$$\deg_{G_1}(x_j) \geq 3, \quad \deg_{G_2}(x_j) \geq 3 \quad \text{and} \quad \deg_{G_3}(x_j) \geq 3,$$

which is impossible.

Thus we must have at least two vertices x_j, x_t such that they have degree sets $\{4,3,0,0\}$ or $\{3,3,1,0\}$. We now change our previous vertex notation and without loss of generality set $j = 1$ and $t = 2$.

Case 1. $DS(x_1) = DS(x_2) = \{4,3,0,0\}$. But then the edge x_1x_2 must have in G_1 or G_2 both endvertices of degree at least three, which is impossible.

Case 2. $\mathrm{DS}(x_1) = \mathrm{DS}(x_2) = \{3,3,1,0\}$. Let x_3, x_4, x_5, x_6 be the vertices having degree four in one of the graphs G_i. We have exhausted two 0's and two 1's in the degree sets, hence we can assume without loss of generality that $\mathrm{DS}(x_3) = \mathrm{DS}(x_4) = \{4,1,1,1\}$ and $\mathrm{DS}(x_5) = \mathrm{DS}(x_6) = \{4,2,1,0\}$. This forces $\mathrm{DS}(x_7) = \mathrm{DS}(x_8) = \{2,2,2,1\}$. But then because x_8 is in each G_i of degree one or two, the other endvertex of every edge x_8x_j must be of degree three or more. This is impossible, since x_7 is always of degree at most two.

Case 3. $\mathrm{DS}(x_1) = \{4,3,0,0\}$ and $\mathrm{DS}(x_2) = \{3,3,1,0\}$. Let x_3, x_4, x_5 be the vertices having degree four in one of the graphs $G_i, i \neq 1$. We have exhausted three of the four 0's and one of the twelve 1's in the degree sets, hence we can assume without loss of generality $\mathrm{DS}(x_3) = \mathrm{DS}(x_4) = \{4,1,1,1\}$ and $\mathrm{DS}(x_5) = \{4,2,1,0\}$ or $\mathrm{DS}(x_5) = \{4,1,1,1\}$. Let x_6 be the vertex having degree three in its degree set. If $\mathrm{DS}(x_5) = \{4,2,1,0\}$, we must have $\mathrm{DS}(x_6) = \{3,2,1,1\}$ (since all 0's have been already assigned). If $\mathrm{DS}(x_5) = \{4,1,1,1\}$, we must have $\mathrm{DS}(x_6) = \{3,2,2,0\}$. That is because the only other unassigned degrees now are 2's and 1's, and we cannot have such a set containing zero that would sum up to seven. In either case we are only left with unassigned 2's and 1's, and again have $\mathrm{DS}(x_7) = \mathrm{DS}(x_8) = \{2,2,2,1\}$, which was shown to be impossible in Case 2.

Now we look at the graph $B40$. The degree sequence of $B40$ is $(5,2,2,2,1,1,1,0)$ and we must have $\mathrm{vds}(x_1) = (5,2,0,0)$ or $(5,1,1,0)$. But we have only four 0's available for all four graph degree sequences, and if x_1 is of degree 0 in two of them, there must be a vertex x_j with a 5 but no 0 in $\mathrm{vds}(x_j)$. However, it is obvious that every vertex x_j, which is of degree five in one of the graphs G_i must be of degree zero in another copy, otherwise the sum of its degrees would exceed seven. Hence, we must have $\mathrm{vds}(x_1) = (5,1,1,0)$ and there are three other vertices whose vertex degree sequence is a permutation of $5,1,1,0$.

Consequently, there are four vertices with their degree sequences consisting of $2,2,2,1$ in some order. Let these vertices be y_1, y_2, y_3, y_4. In each G_i, we have one of the vertices y_j adjacent to two other vertices y_s and y_t. But then the subgraph of K_8 induced on y_1, y_2, y_3, y_4 would contain at least eight edges, which is absurd. $\square$

5 Decompositions of K_7 and K_8

We now present decompositions of K_7 into graphs $B41$–$B45$, $B47$, and $B48$.

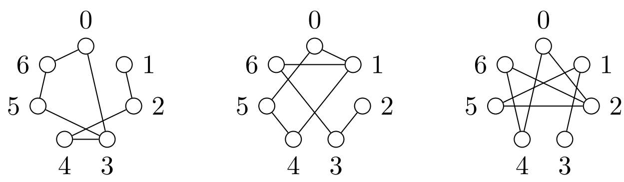

Figure 2: $B41$-decomposition of K_7

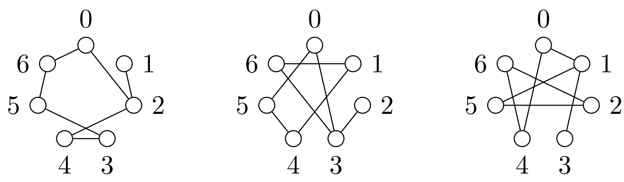

Figure 3: $B42$-decomposition of K_7

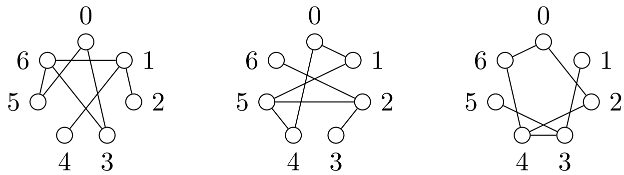

Figure 4: $B43$-decomposition of K_7

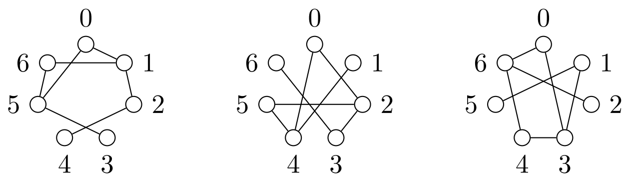

Figure 5: $B44$-decomposition of K_7

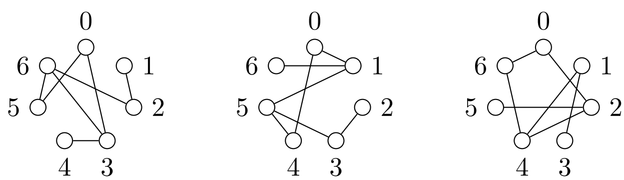

Figure 6: $B45$-decomposition of K_7

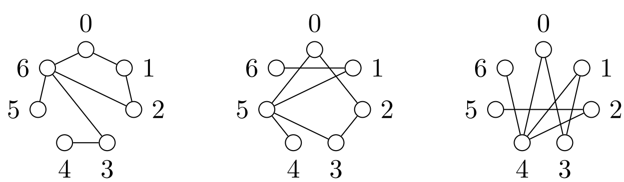

Figure 7: $B47$-decomposition of K_7

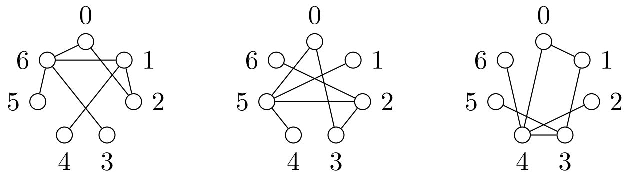

Figure 8: $B48$-decomposition of K_7

The following observation arises directly from the figures above.

Observation 5.1. *There exists a G-decomposition of K_7 where G is any of $B41$–$B45, B47, B48$.*

Next we present packings of K_7 with graphs $G-e$ for the graphs that do not decompose K_7, that is, $B39, B40$, and $B46$. These packings will be later used for decompositions of graphs $K_{14} - K_7$. The dashed edge is shown to illustrate the whole graph G including the edge e.

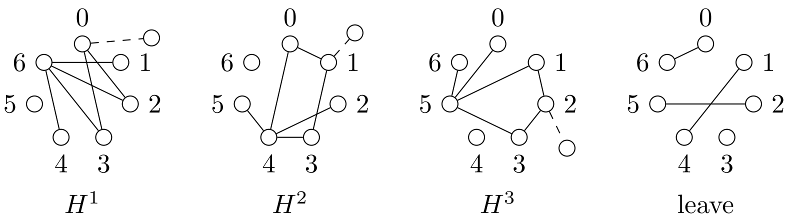

Figure 9: Packing of K_7 with $B39 - e$

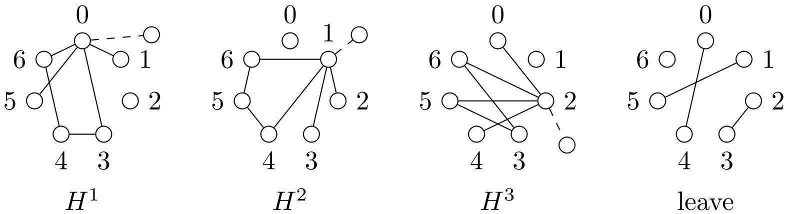

Figure 10: Packing of K_7 with $B40 - e$

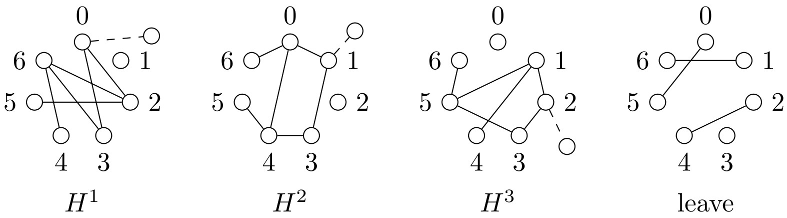

Figure 11: Packing of K_7 with $B46 - e$

Now we show decompositions of K_8 into $B41$–$B48$. The decompositions into $B41, B43, B44$, and $B47$ were provided by Meszka [22]. Recall that decompositions of K_8 into $B39$ and $B40$ do not exist.

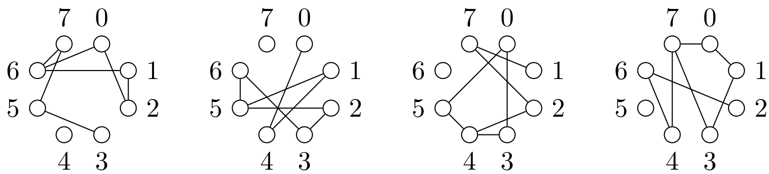

Figure 12: $B41$-decomposition of K_8

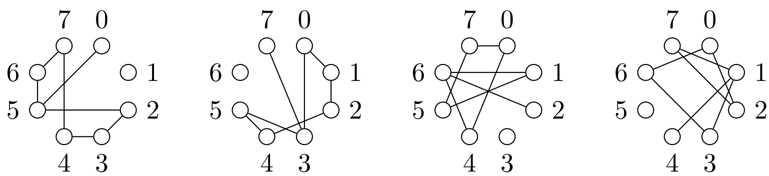

Figure 13: $B42$-decomposition of K_8

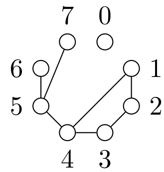

Figure 14: $B43$-decomposition of K_8

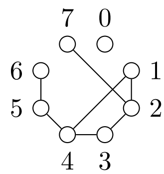

Figure 15: $B44$-decomposition of K_8

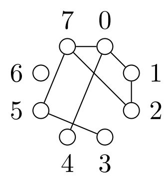

Figure 16: $B45$-decomposition of K_8

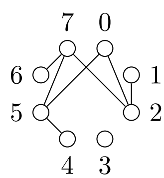

Figure 17: $B46$-decomposition of K_8

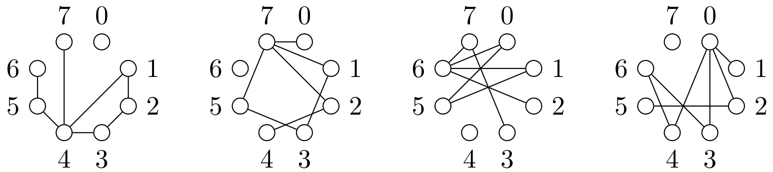

Figure 18: $B47$-decomposition of K_8

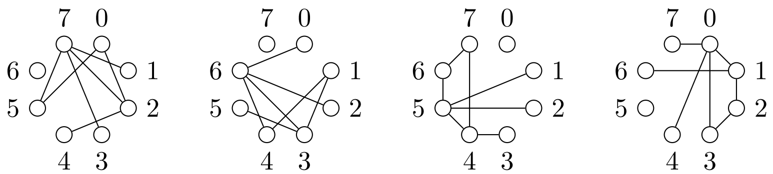

Figure 19: $B48$-decomposition of K_8

Observation 5.2. *There exists a G-decomposition of K_8 where G is any of $B41$–$B48$.*

6 Decompositions of K_n for $n \equiv 0, 1 (\text{mod } 14)$

All decompositions of K_n for $n \equiv 1 \pmod{14}$ are based on α-labelings of the respective graphs.

The labelings we use can be easily modified to 1-rotational σ^+-labelings by replacing the label 7 with ∞ except for graph $B42$, where a 1-rotational σ^+-labeling does not exist. We present the labelings in Figures 20–23.

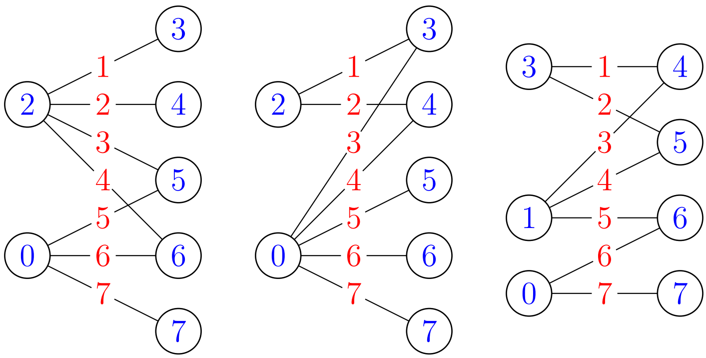

Figure 20: σ^+-labelings of $B39, B40, B41$ (left to right)

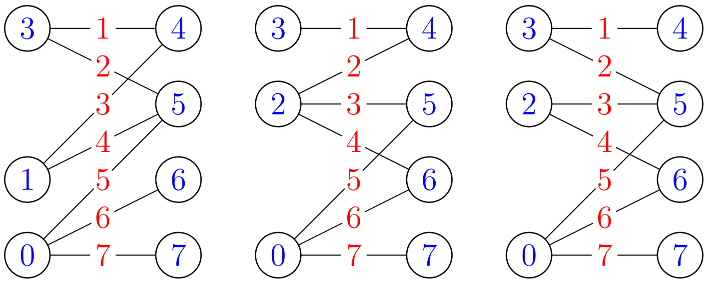

Figure 21: σ^+-labelings of $B43, B44, B45$ (left to right)

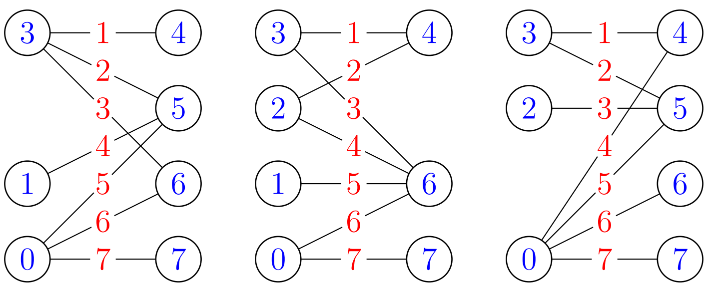

Figure 22: σ^+-labelings of $B46, B47, B48$ (left to right)

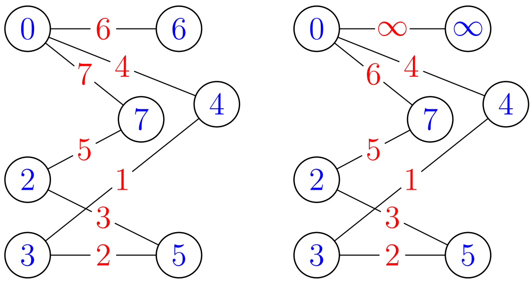

Figure 23: α-labeling (left) and 1-rotational ρ-labeling (right) of $B42$

The result for $n \equiv 1 \pmod{14}$ now follows immediately from Theorem 2.10.

Theorem 6.1. *There exists a G-decomposition of K_n for all $G \cong B39$, $B40$, ..., $B48$ and every $n \equiv 1 \pmod{14}$.*

For $n \equiv 0 \pmod{14}$, we first present a $B42$-decomposition of K_{14k}.

Lemma 6.2. *There exists a $B42$-decomposition of K_{14k} for any $k \geq 1$.*

Proof. As shown in Figure 23, there exist both α-labeling and 1-rotational ρ-labeling of the graph $B42$. Then by Corollary 2.16 and Theorem 2.7 there exists a $B42$-decomposition of $K_{14,14}$ and K_{14}, respectively.

For every $k \geq 2$, the graph K_{14k} can be decomposed into k copies of K_{14} and $\binom{k}{2}$ copies of $K_{14,14}$. The conclusion follows. $\square$

Now, the result for $n \equiv 0 \pmod{14}$ follows from Theorem 2.13 and Lemma 6.2.

Theorem 6.3. *There exists a G-decomposition of K_n for $G \cong B39, B40, \ldots, B48$ and every $n \equiv 0 \pmod{14}$.*

7 Decompositions of K_n for $n \equiv 7 \pmod{14}$

To decompose complete graphs with $n = 14k + 7$ vertices, we mostly use our previous results. We first break up the graph K_{14k+7} into K_{14k}, K_7, and $2k$ copies of $K_{7,7}$. Then we decompose each of them separately, using labelings or decompositions provided in previous sections. For the three graphs that do not decompose K_7, namely $B39, B40$ and $B46$, we split K_{14k+7} into K_{14k}, $2k-1$ copies of $K_{7,7}$ and one copy of $K_{14} - K_7$. In this case, we need to find a G-decomposition of $K_{14} - K_7$ for $G \cong B39, B40$ and $B46$.

Decompositions of K_{14k} were shown in Section 6. Now we formally prove the existence of decompositions of $K_{7,7}$. Although the Lemma in fact follows from Corollary 2.16, we prefer to show the details and introduce some notation that will be later useful for decompositions of graphs $K_{14} - K_7$.

Lemma 7.1. *Each graph $G \cong B39, B40, \ldots, B48$ decomposes the complete bipartite graph $K_{7,7}$.*

Proof. We take the α-labelings f shown in Figures 20–23 and transform them into α-bilabelings f' by setting $f'(u) = f(u)$ for every $u \in U$ (always shown in the left column) and $f'(v) = f(v) - 1$ for every $v \in V$ (always in the right column). For clarity, we identify the vertices with their labels and add subscript 1 to labels of vertices in U and subscript 2 to labels in V. The graph $K_{7,7}$ then has vertex sets $U = \{0_1, 1_1, \ldots, 6_1\}$ and $V = \{0_2, 1_2, \ldots, 6_2\}$.

The graph obtained this way is the copy G^1 of G. Then for $i = 2, 3, \ldots, 7$ if in G^1 we have $f'(u) = a_1$, in G^i the label of u will be $(a+i-1)_1$ and similarly when $f'(v) = b_2$, in G^i the label will be $(b+i-1)_2$. Because each copy G^1 contains exactly one edge of each length $0, 1, \ldots, 6$, the collection $G^1, G^2, \ldots, G^7$ forms a decomposition of $K_{7,7}$. $\square$

To illustrate the Lemma 7.1 method, we present in Figure 24 the first four copies of B39.

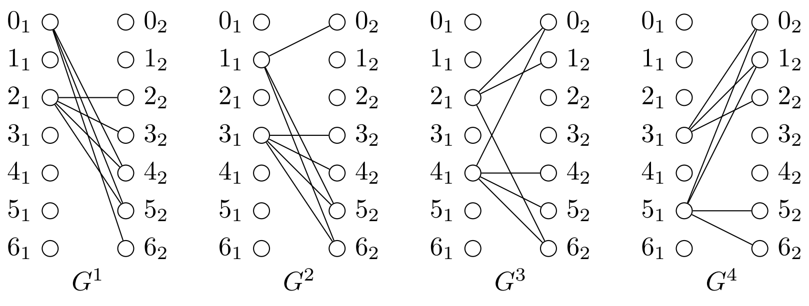

Figure 24: The first four copies of a $B39$-decomposition of $K_{7,7}$

To decompose $K_{14} - K_7$ into graphs $B39, B40$ and $B46$, we proceed as follows. We always "borrow" one edge from G^i for $i = 1, 2, 3$ and add it (shown dashed) to the corresponding graph $G - e = H^i$ shown in Figures 9–11 to produce a copy $\widetilde{H}^i$. Then we replace the borrowed edge with the appropriate excess edge from the packing leave (dash-dot) to get the copy $\widetilde{G}^i$.

Lemma 7.2. *The graphs* $B39, B40$ *and* $B46$ *decompose* $K_{14} - K_7$*, the complete graph* K_{14} *with a hole of size* 7.

Proof. For each of the three graphs, the copies $\widetilde{H}^1, \widetilde{H}^2, \widetilde{H}^3, \widetilde{G}^1, \widetilde{G}^2, \widetilde{G}^3$ and G^4 are shown in Figures 25, 26 and 27, respectively.

The remaining copies G^5, G^6 and G^7 arise from the α-bilabelings described in the proof of Lemma 7.1. □

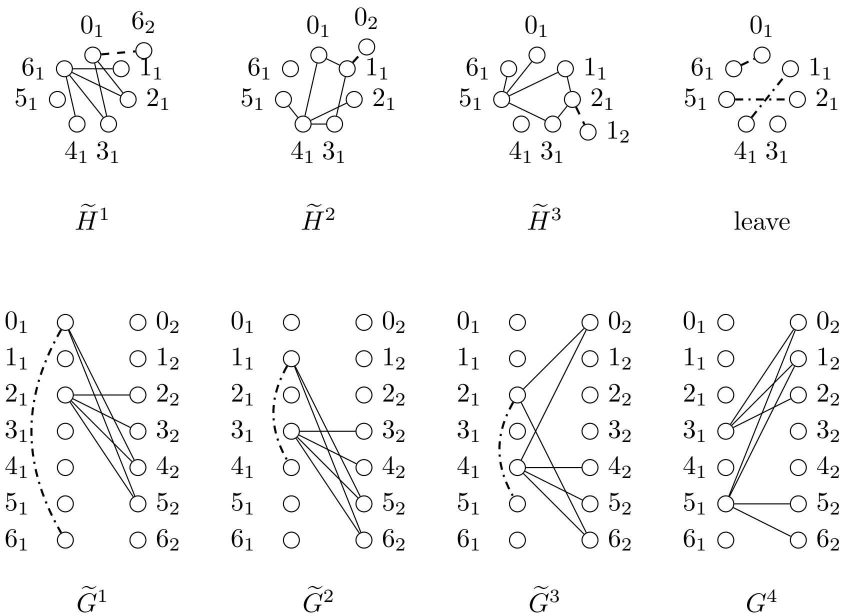

Figure 25: Copies $\widetilde{H}^i$ and $\widetilde{G}^i$ of $B39$ in $K_{14} - K_7$

The main result of this section now follows easily.

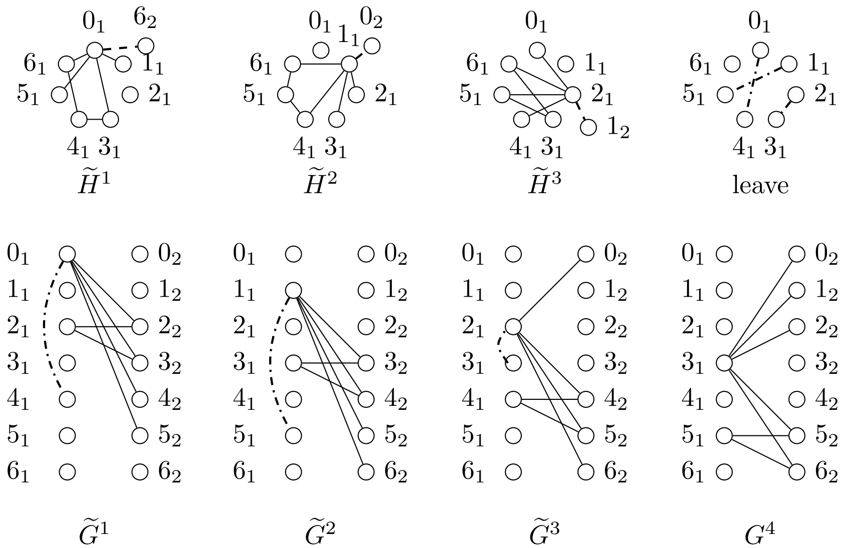

Figure 26: Copies $\widetilde{H}^i$ and $\widetilde{G}^i$ of $B40$ in $K_{14} - K_7$

Theorem 7.3. *There exists a G-decomposition of K_n for all $G \cong B39$, $B40, \ldots, B48$ and every $n \equiv 7 \pmod{14}$ except when $n = 7$ and $G \cong B39$, $B40$ or $B46$.*

8 Decompositions of K_n for $n \equiv 8 \pmod{14}$

To decompose complete graphs with $n = 14k + 8$ vertices, we again use our previous results. For $k = 0$, the results follow directly from the previous sections. For $k \geq 1$, we split the graph K_{14k+8} into several subgraphs first, then decompose each of them separately. The ingredients will include $K_{14k}, K_{14k-7}, K_{15}, K_{14} - K_7, K_{8,7}$, and $K_{7,7}$.

Decompositions of K_{14k} and K_{15} were shown in Section 6, and those of $K_{14k-7}, K_{14} - K_7$ and $K_{7,7}$ in Section 7.

Therefore, we only need to show decompositions of $K_{8,7}$. Because no established labeling techniques can be used here, we need to find ad hoc decompositions. One can see that if G decomposes $K_{2,7}$ or $K_{4,7}$, then it

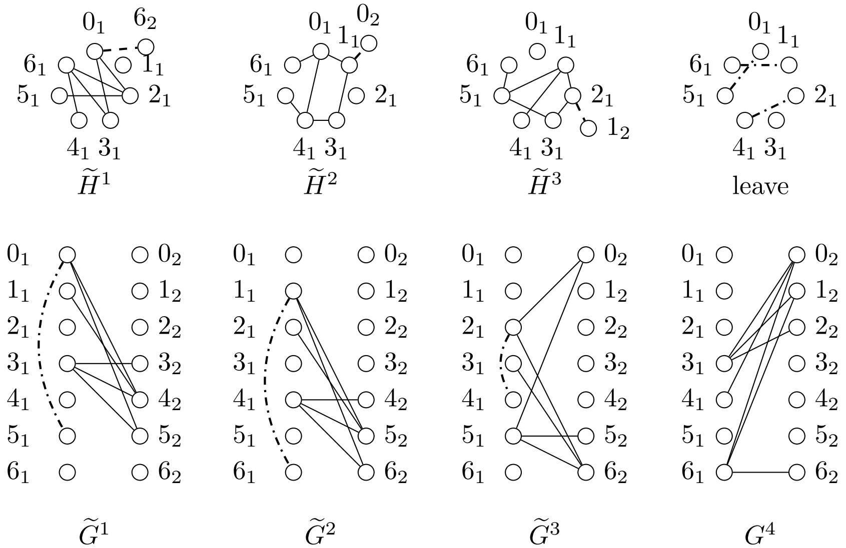

Figure 27: Copies $\widetilde{H}^i$ and $\widetilde{G}^i$ of $B46$ in $K_{14} - K_7$

also decomposes $K_{8,7}$. We were able to find decompositions of $K_{2,7}$ or $K_{4,7}$ in most cases, only graph $B43$ required decompositions of $K_{4,14}$. The decompositions are presented in the figures below.

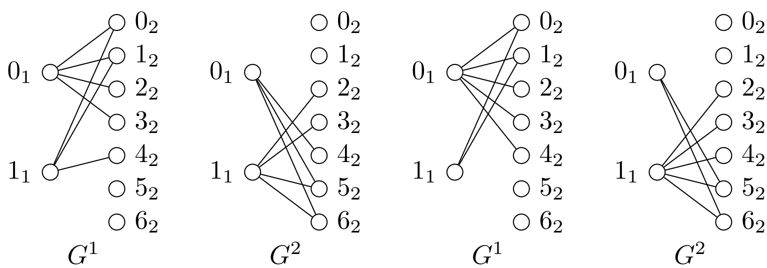

Figure 28: Decompositions of $K_{2,7}$ into $B39$ (left) and $B40$ (right)

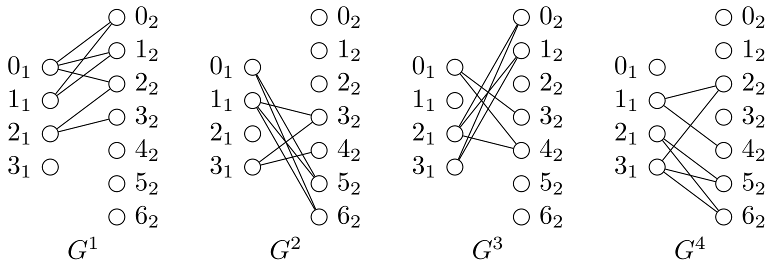

Figure 29: Decomposition of $K_{4,7}$ into $B41$

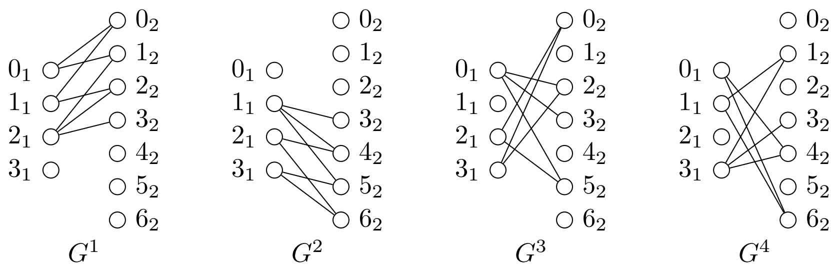

Figure 30: Decomposition of $K_{4,7}$ into $B42$

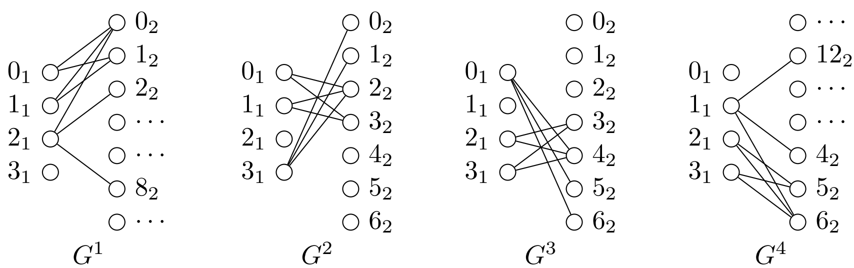

Figure 31: Decomposition of $K_{4,14}$ into $B43$

Notice that for $B43$ we actually had to decompose $K_{4,14}$. We present first four copies, the remaining four can be obtained by increasing each of the labels b_2 to $(b+7)_2$, taken modulo 14. Observe that the edges $2_1 1_2$ and

$1_1 5_2$ are the only edges $a_1 b_2$ for $a = 0, 1, 2, 3$ and $b = 0, 1, \ldots, 6$ not used in the first four copies, but are replaced by edges $2_1 8_2$ and $1_1 12_2$ instead.

Next we present decompositions of $K_{4,7}$ into graphs $B44$–$B48$.

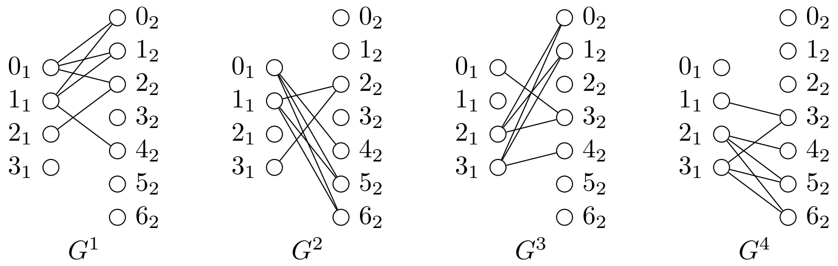

Figure 32: Decomposition of $K_{4,7}$ into $B44$

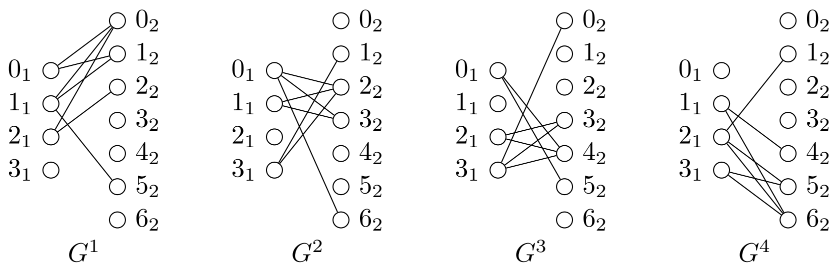

Figure 33: Decomposition of $K_{4,7}$ into $B45$

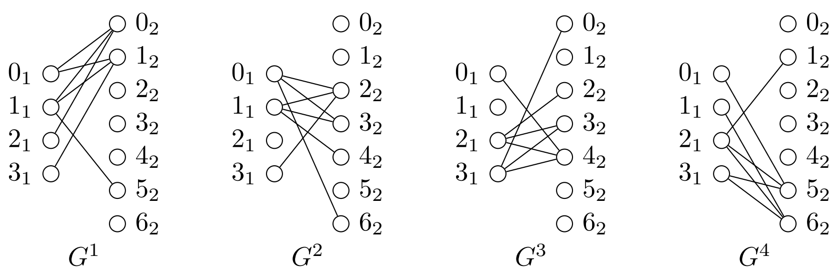

Figure 34: Decomposition of $K_{4,7}$ into $B46$

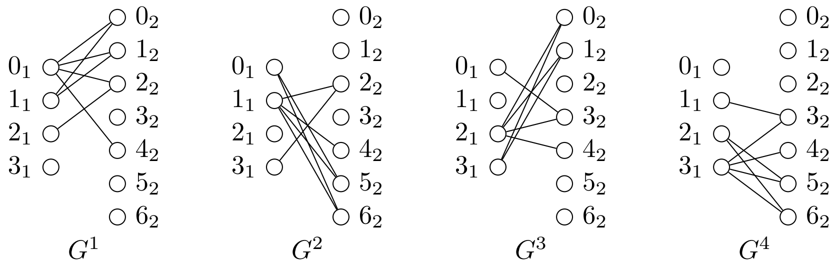

Figure 35: Decomposition of $K_{4,7}$ into $B47$

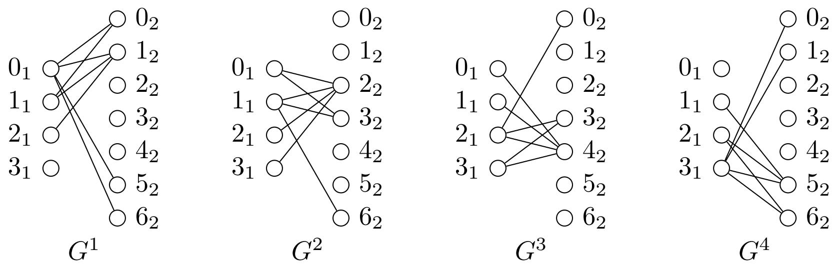

Figure 36: Decomposition of $K_{4,7}$ into $B48$

From the decompositions shown in Figures 28–36, we immediately have the following.

Lemma 8.1. *The graphs $B39, \dots, B42$ and $B44, \dots, B48$ decompose the complete bipartite graph $K_{8,7}$; the graph $B43$ decomposes $K_{8,14}$.*

Now we can prove the main result of this section.

Theorem 8.2. *There exists a G-decomposition of K_n for all $G \cong B39$, $B40$, ..., $B48$ and every $n \equiv 8 \pmod{14}$ except when $n = 8$ and $G \cong B39$ or $B40$.*

Proof. Decompositions of K_8 for $B41, B42, \dots, B48$ follow from Observation 5.1, and non-existence for $n = 8$ and $G \cong B39, B40$ follows from Proposition 4.2. Hence in what follows, we always assume that $k \geq 1$.

For $B39$ and $B40$, when $k = 1$, we break up K_{22} into K_{15}, $K_{14} - K_7$ and $K_{8,7}$. Their decompositions into $B39$ and $B40$ exist by Theorem 6.1 and Lemmas 7.2 and 8.1.

When $k > 1$, we break up K_{14k+8} into K_{15}, K_{14k-7}, and $2k-1$ copies of each $K_{7,7}$ and $K_{8,7}$. Their decompositions exist by Theorems 6.1 and 7.3, and Lemmas 7.1 and 8.1.

For $B43$, we split up K_{14k+8} into K_8, K_{14k}, and k copies of $K_{8,14}$, and we have $B43$-decompositions by Observation 5.1, Theorem 6.3, and Lemma 8.1.

For the remaining graphs, we split K_{14k+8} into K_8, K_{14k}, and $2k$ copies of each $K_{7,7}$ and $K_{8,7}$, the desired decompositions of which exist by Observation 5.2, Theorem 6.3 and Lemmas 7.1 and 8.1. □

9 Conclusion

Our main result now follows directly from the obvious necessary condition, Propositions 4.1 and 4.2, and Theorems 6.1, 6.3, 7.3, and 8.2.

Theorem 9.1. *There exists a G-decomposition of K_n for $G \cong B39$, $B40$, $\ldots$, $B48$ if and only if $n \equiv 0, 1 \pmod 7$ except when $n = 7$ and $G \cong B39$, $B40$ or $B46$; or when $n = 8$ and $G \cong B39$ or $B40$.*

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