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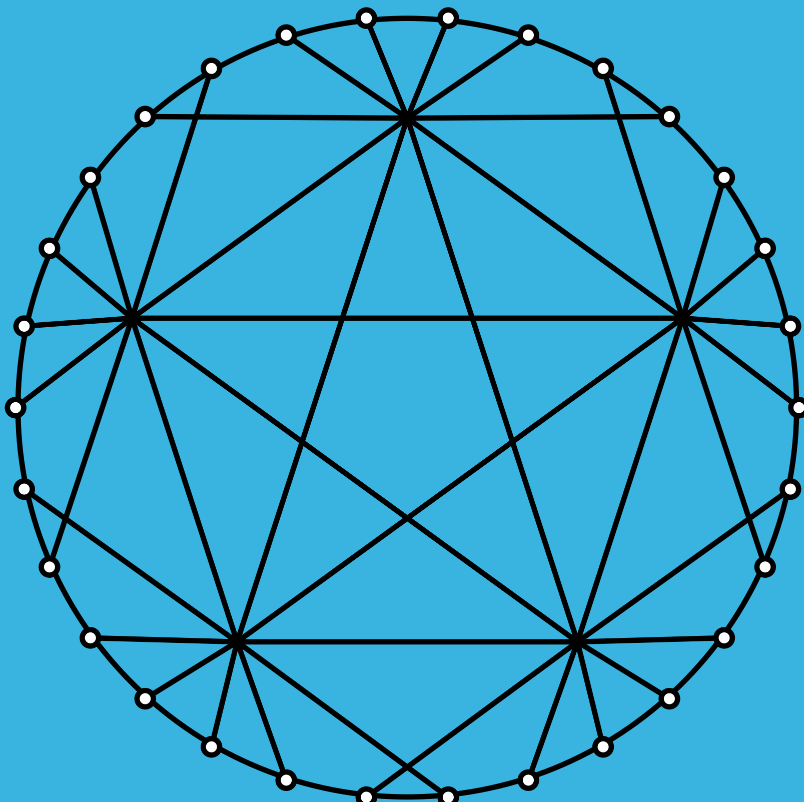

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Honeycomb Toroidal Graphs

Brian Alspach

University of Newcastle, Callaghan, Australia
brian.alspach@newcastle.edu.au

Abstract

Honeycomb toroidal graphs are bipartite trivalent Cayley graphs on generalized dihedral groups. We examine the two historical threads leading to these graphs, some of the properties that have been established, and some open problems.

1 Introduction

This paper discusses a family of graphs called *honeycomb toroidal graphs*. They have arisen in two distinct settings which are discussed in the next two sections. This is followed by an examination of some of their properties and some parameters of interest. Several open problems also are mentioned.

Given the disparate subject areas employing graphs as models, there are a variety of concepts for which different terms are used across disciplines. Thus, we shall mention some terminology used in this paper. A *graph* has neither loops nor multiple edges. The *valency* of a vertex v, denoted $\mathrm{val}(v)$, is the number of edges incident with v. The *order* of a graph is the cardinality of its vertex set and the *size* of a graph is the cardinality of its edge set.

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A *path of length* ℓ in a graph is a subgraph consisting of a sequence

$$v_0, v_1, \ldots, v_\ell$$

of $\ell+1$ distinct vertices such that the edge v_iv_{i+1} belongs to the path for $i = 0, 1, \ldots, \ell-1$. A *cycle of length* ℓ is a connected subgraph of size ℓ in which every vertex has valency 2. Cycles are denoted by a sequence of vertices as they occur along the cycle with the convention that the first vertex and the last vertex of the sequence are the same in order to distinguish it from a path. A *Hamilton cycle* in a graph is a cycle containing every vertex of the graph, and a *Hamilton path* is a path containing every vertex.

2 Algebraic And Topological Viewpoint

An *equivelar* map with Schläfli type (a, b) has all face boundaries of length a and every vertex of valency b. Altshuler [5] considered three families of equivelar maps on the torus and was able to show that every graph in two of the families possesses a Hamilton cycle, but was unable to do so for the other family. The latter family consists of the equivelar maps with Schläfli type $(6, 3)$. Many of these graphs, but not all, are Cayley graphs on the appropriate dihedral group. So this problem arising in topological graph theory impinges on another problem which has drawn considerable attention for fifty years, namely, does every connected Cayley graph of order at least three have a Hamilton cycle?

The answer to the preceding question for Cayley graphs on abelian groups was known to be yes as early as the first edition of Lovász's book entitled *Combinatorial Problems and Exercises* [12]. However, a much stronger result by Chen and Quimpo [7] appeared in 1981. Their theorem follows two definitions. A graph X is *Hamilton-connected* if for every pair of vertices u and v in X there is a Hamilton path whose terminal vertices are u and v. A bipartite graph X is *Hamilton-laceable* if the same property holds for any two vertices in opposite parts.

Theorem 2.1. [7] *If X is a connected Cayley graph of valency at least 3 on an abelian group, then X is Hamilton-connected unless it is bipartite in which case it is Hamilton-laceable.*

Note that the preceding theorem implies that every edge of a connected Cayley graph on an abelian group belongs to a Hamilton cycle. If the valency is at least 3, it is implied by the theorem. If the valency is 2, the graph is a Hamilton cycle.

The dihedral group is close to being abelian in the sense that the dihedral group D_n of order $2n$ contains an abelian subgroup of order n. It still is not known whether every connected Cayley graph on D_n is hamiltonian in spite of the efforts of a non-trivial number of people working on the problem for the last forty plus years.

As we shall see soon, when considering Cayley graphs on dihedral groups, those for which the connection set consists of three reflections turn out to be crucial. Let's now take a closer look at these particular graphs.

Throughout this paper we let D_n denote the dihedral group of degree n and order $2n$. We visualize the group as the symmetries of a regular n-gon. So the group is generated by an element ρ of order n (it rotates the n-gon cyclically) and a reflection τ. Thus, $|\tau| = 2$ and $\tau\rho\tau = \rho^{-1}$. The cyclic subgroup $\langle\rho\rangle$ has index 2 in D_n. Note that the coset $\langle\rho\rangle\tau$ consists of n reflections. When n is odd, the n reflections are the only involutions in D_n, whereas, $\rho^{n/2}$ also is an involution when n is even.

We are interested in Cayley graphs on D_n whose connection sets consist of three reflections. Let $S = \{\rho^\alpha\tau, \rho^\beta\tau, \rho^\gamma\tau\}$, where $0 \le \alpha < \beta < \gamma < n$. It is clear that the connection set $\{\tau, \rho^{\beta-\alpha}\tau, \rho^{\gamma-\alpha}\tau\}$ produces an isomorphic Cayley graph. Hence, we shall assume the connection set has the form $S = \{\tau, \rho^x\tau, \rho^y\tau\}$, where $0 < x < y < n$.

The graph $X = \mathrm{Cay}(D_n; S)$ is connected if and only if it is the case that $\gcd(n, x, y) = 1$. So dealing with connectivity is straightforward. The subgraph Y generated using just τ and $\rho^x\tau$ consists of m components each of which is a cycle of length $r = 2n/m$, where $m = \gcd(n, x)$. The case in which we are most interested is when $m > 1$ and X is connected. This means that the element $\rho^y\tau$ generates edges that connect the m components of Y to form a single component for X. We want to take a careful look at these graphs to see how to represent them nicely.

The vertices of $\langle\rho\rangle$ are cyclically labelled $1, \rho, \rho^2, \ldots, \rho^{n-1}$ and those of $\langle\rho\rangle\tau$ are cyclically labelled $\tau, \rho\tau, \rho^2\tau, \ldots, \rho^{n-1}\tau$. The vertices of each component alternate between belonging to $\langle\rho\rangle$ and $\langle\rho\rangle\tau$. Thus, the cycle length r is even and $r \ge 4$.

Let $\rho^y\tau$ generate an edge joining a vertex ρ^a of $\langle\rho\rangle$ in a cycle C_1 of Y to the vertex $\rho^{a+y}\tau$ of $\langle\rho\rangle\tau$ in another cycle C_2. Then $[\rho^a, \rho^a\tau, \rho^{a-x}]$ is a subpath of length 2 in C_1, that is, ρ^a and ρ^{a-x} are distance 2 apart on the cycle C_1. Similarly, $\rho^{a+y}\tau$ and $\rho^{a+y-x}\tau$ are distance 2 apart on the cycle C_2. Note that $\rho^y\tau$ generates an edge from ρ^{a-x} to $\rho^{a+y-x}\tau$. Hence, we see that $\rho^y\tau$ joins the two cycles C_1 and C_2 by joining pairs of alternate vertices along the two cycles.

Therefore, we may label the vertices of the graph as $u_{i,j}$, $0 \leq i \leq m-1$ and $0 \leq j \leq r-1$ so that as we label the columns from left to right, it is clear that we may assure that the edges between successive columns have the same second coordinate. However, once the last column is labelled the only feature we know about the edges from the last column back to the first column is that they have the same change in the second coordinate, that is, they have the same *jump*.

We now have a straightforward description of these graphs. They are called *honeycomb toroidal graphs* and are denoted $\mathrm{HTG}(m, r, \ell)$, where m is the number of column cycles, r is the length of the column cycles so that $r \geq 4$ and is even, and ℓ is the jump from the last column back to the first. Following is a description of the edges:

- $u_{i,j}u_{i,j+1}$ for $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, r-1$, where the second subscript is reduced modulo r (these are called *vertical edges*);
- $u_{i,j}u_{i+1,j}$ for $i = 0, 1, \ldots, m-2$ and all j such that $i+j$ is odd (these are called *flat edges*; and
- $u_{m-1,j}u_{0,j+\ell}$, where m, j, ℓ all have the same parity (these are called *jump edges*.

Even though this family of graphs has arisen from certain Cayley graphs on dihedral groups, we should point that not all of them are Cayley graphs on dihedral groups. In [4] it is shown that all of them are Cayley graphs on generalized dihedral groups.

Figure 1 demonstrates clearly how $\mathrm{HTG}(m, r, \ell)$ may be embedded on a torus for any choice of the parameters. Even though these graphs all have nice embeddings on the torus, they are slightly misnamed in that they are not all toroidal graphs. This turn of events comes about because in order for a graph that embeds on the torus to be toroidal it must be non-planar. When $m = 1$, the graphs $\mathrm{HTG}(1, 4s, 3)$ are planar for $s > 1$. The graphs $\mathrm{HTG}(2, r, 0)$, for all even $r \geq 4$, are planar (note that $\mathrm{HTG}(2, 4, 2)$ is isomorphic to $\mathrm{HTG}(2, 4, 0)$) graphs. It is not hard to see that all others

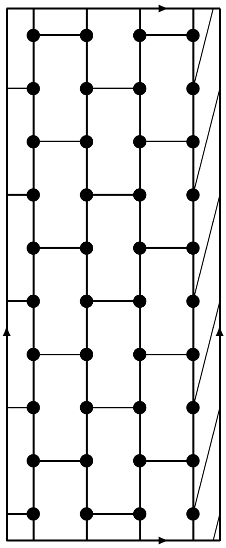

Figure 1: HTG$(4, 10, 2)$ embedded on the torus

are non-planar. Nevertheless, we shall refer to all graphs in the family as honeycomb toroidal graphs.

3 Network Topology Viewpoint

Network topology refers to methods used to connect objects together to perform certain tasks. For example, connecting computers together to form a computer network or connecting processors within a single computer fall within the area. Some desirable properties are small valency so that the number of direct connections is not too big and symmetry meaning that all the vertices are essentially the same which allows local algorithms to be the same at each vertex.

One approach is to start with tesselations of the plane by regular polygons. These have an infinite number of vertices so that some modifications are required. One such modification is to bound a finite region of a tesselation with a "nice polygon" to obtain a finite graph. The latter graph is called a *mesh*. The mesh is not regular but the addition of a few edges may result in a graph that is not only regular but also is vertex-transitive.

The tesselation of the plane by regular hexagons is one source for which this was done. Stojmenovic [17] suggested three bounding types of polygons to obtain meshes: a hexagonal polygon, a square polygon and a rhombic polygon. He then determined ways to add edges so that all vertices have valency 3 and the resulting graph is vertex-transitive.

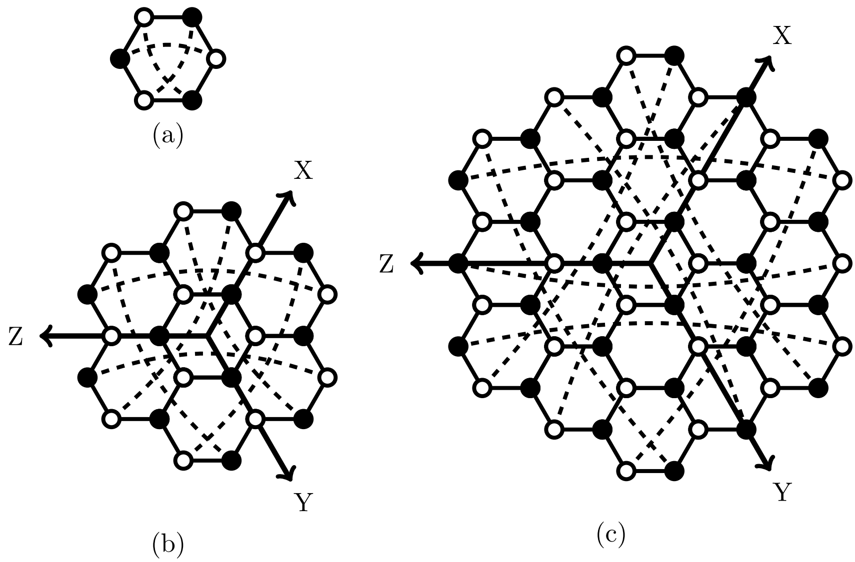

Figure 2

Figure 2 shows the three smallest graphs obtained by Stojmenovic using a hexagonal bounding polygon. We need to examine the viewpoint in some detail because this formed the foundation for the way subsequent researchers in the area developed the ideas. He called the graph in Figure 2(a) a *hexagonal torus of size 1.* The graph in Figure 2(b) he called the hexagonal torus of size 2. Thus, to increase the size by 1 we add a ring of hexagons around the current graph. This is a very geometric way of building the graphs.

His use of a square as a bounding polygon has since been extended to using a rectangle, and his use of a rhombus for a bounding polygon has been extended to using a parallelogram. All three types of graphs share the property that they arise geometrically. They are, in fact, very special honeycomb toroidal graphs as domonstrated in the next proposition which is a summary of results in [9].

Proposition 3.1. *The hexagonal torus of size* m *is* $\mathrm{HTG}(m, 6m, 3m)$ *for* $m \geq 1$. *The rectangular torus is* $\mathrm{HTG}(m, n, 0)$ *for even* $m \geq 2$. *The parallelogramic torus is* $\mathrm{HTG}(m, n, m')$, *where* $m' \equiv m (\mathrm{mod}\ n)$ *and* $0 \leq m' < m$.

Some comments about terminology are in order. Because of the way the network topology approach developed these graphs, they were understandably viewed as special. Thus, when it was discovered [9] how to broaden the construction, the term *generalized honeycomb torus* was adapted and appears in many papers. However, we object to this terminology for two reasons leading to the term honeycomb toroidal graph being used.

The first objection is because the torus is a closed orientable surface of genus one and even though these graphs have nice embeddings on the torus, the graphs themselves should not be called tori. The second objection arises because we have seen that the only differences between them come from changing three descriptive parameters. There is no particular set of parameters that is special and the term ‘generalized’ is inappropriate.

4 Hamiltonicity

Hamiltonicity refers to various properties of graphs revolving around Hamilton paths and Hamilton cycles. We consider two properties in this section. The first is the hamiltonian property, that is, does $\mathrm{HTG}(m, r, \ell)$ have a Hamilton cycle? The second property is Hamilton laceability, that is, is every honeycomb toroidal graph Hamilton-laceable?

The answer to the first question is yes and was proven in [20]. We give a short proof of this result but before doing so we discuss a useful constructive technique for honeycomb toroidal graphs.

Consider three consecutive columns of $\mathrm{HTG}(m, r, \ell)$ subscripted by $i, i+1, i+2$, and the flat edges $u_{i,t_1}u_{i+1,t_1}; u_{i,t_2}u_{i+1,t_2}; \ldots; u_{i,t_k}u_{i+1,t_k}$, where $0 \leq t_1 < t_2 < \cdots < t_k < r$. Extend each edge $u_{i,t_a}u_{i+1,t_a}$ to a path from u_{i,t_a} to u_{i+2,t_a} by adding the vertical path from u_{i+1,t_a} down to $u_{i+1,1+t_{a-1}}$ followed by the edge $u_{i+1,1+t_{a-1}}u_{i+2,1+t_{a-1}}$ and then back up column $i+2$

to u_{i+2,t_a}. We then obtain paths from u_{i,t_a} to u_{i+2,t_a} that use all the vertices of columns $i+1$ and $i+2$. This operation is called the *vertical downward fill* for columns $i+1$ and $i+2$. The *vertical upward fill* is defined in an obvious analogous manner. These operations are most clearly seen by looking at Figure 3 which shows an example of both vertical fills and makes everything obvious. Note that this technique may be applied to graphs other than honeycomb toroidal graphs.

Theorem 4.1. *Every honeycomb toroidal graph is hamiltonian.*

Proof. Claim: If $\mathrm{HTG}(m, r, \ell)$ is hamiltonian, then $\mathrm{HTG}(m+2, r, \ell)$ also is hamiltonian. It is easy to see that there must be at least one flat edge between column 0 and column 1 in any Hamilton cycle of $\mathrm{HTG}(m, r, \ell)$ when $m \geq 2$. So let $u_{0,j_1}u_{1,j_1}; u_{0,j_2}u_{1,j_2}; \ldots; u_{0,j_t}u_{1,j_t}$, $0 < j_1 < j_2 < \cdots < j_t < r$, be the flat edges between column 0 and column 1 in some Hamilton cycle of $\mathrm{HTG}(m, r, \ell)$.

Subdivide each flat edge with two new vertices in each edge. Remove the central edge in each of the subdivided edges and use vertical fills between the two new columns to obtain a Hamilton cycle in $\mathrm{HTG}(m+2, r, \ell)$.

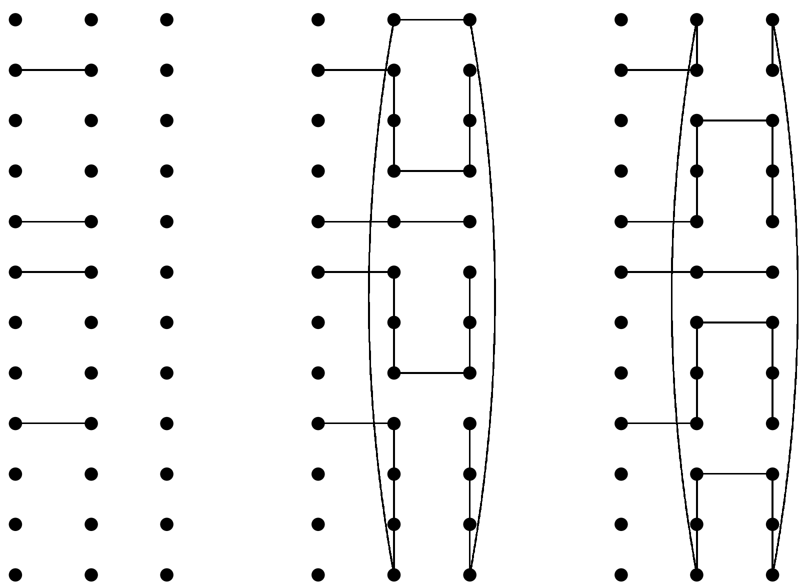

Figure 3

Thus, it suffices to prove that HTG$(2, r, \ell)$ and HTG$(3, r, \ell)$ are hamiltonian. Consider $m = 2$ first. For each even i, let P_i be the 4-path

$$u_{0,i}u_{0,i+1}u_{1,i+1}u_{1,i}u_{0,i+\ell}.$$

Start a path with P_0 followed by P_ℓ followed by $P_{2\ell}$ and so on. This eventually closes off to form a cycle. If the cycle is a Hamilton cycle, we are done. If it is not a Hamilton cycle, then perform vertical fills upwards on each flat edge (removing the flat edge) to obtain a Hamilton cycle. (In the special case that $\ell = 0$, it is easy to see how to obtain a Hamilton cycle from the initial collection of 4-cycles.)

To prove that HTG$(3, n, \ell)$ is hamiltonian, we start with HTG$(1, r, \ell)$. The column itself is a Hamilton cycle that uses none of the jump edges. By Smith's Theorem [18] there is a second Hamilton cycle C and it must use some jump edges. Each jump edge has the form $u_{0,i}u_{0,j}$ with i odd and $j = i + \ell$ even. Let $0 < i_1 < i_2 < \cdots < i_t < r$ be the odd subscripted vertices of the jump edges in C. Add two columns whose vertices are labelled conventionally. Replace the jump edge $u_{0,i_q}u_{0,i_q+\ell}$ with the jump edge $u_{2,i_q}u_{0,i_q+\ell}$ and add the flat edge $u_{0,i_q}u_{1,i_q}$ for each $i_1, i_2, \ldots, i_t$. Now use vertical fills between columns 1 and 2 to obtain a Hamilton cycle in HTG$(3, r, \ell)$ completing the proof. □

The second question is not yet settled and we state it as a research problem.

Research Problem 1. Is every HTG(m, r, ℓ) Hamilton-laceable?

Some comments about the preceding problem are in order. It is a significant problem because an affirmative answer implies that the family of connected Cayley graphs of valency at least 3 on generalized dihedral groups satisfies the conclusions of the Chen - Quimpo Theorem. A special conclusion from this, of course, is that every connected Cayley graph on a dihedral group is hamiltonian. The fact that the latter conclusion still is unsettled is a frustrating situation.

There has been some progress on Research Problem 1. In [2] it is proved that HTG(m, r, ℓ) is Hamilton-laceable whenever m is even. This leaves the case that m is odd. A few special cases for $m = 1$ are solved in [3]. The following result is due to McGuinness [13]. His manuscipt contains a long proof and was not published. Consequently, we provide a short proof here for convenience.

Theorem 4.2. *If* $\mathrm{HTG}(1, r, \ell)$ *is Hamilton-laceable, then* $\mathrm{HTG}(m, r, \ell)$ *is Hamilton-laceable for all odd* $m \geq 1$.

Proof. Using a method similar to the proof of Theorem 4.1, it is easy to show that if $\mathrm{HTG}(3, r, \ell)$ is Hamilton-laceable, then $\mathrm{HTG}(m, r, \ell)$ is Hamilton-laceable for all odd $m \geq 3$. This reduces the proof to showing that $\mathrm{HTG}(1, r, \ell)$ being Hamilton-laceable implies that $\mathrm{HTG}(3, r, \ell)$ is Hamilton-laceable.

Assume that $\mathrm{HTG}(1, r, \ell)$ is Hamilton-laceable. Let P' be a Hamilton path in $\mathrm{HTG}(1, r, \ell)$ from $u_{0,0}$ to $u_{0,j}$ using at least one jump edge. Because $\mathrm{HTG}(1, r, \ell)$ is bipartite, j must be odd and the subscripts of the end vertices of jump edges have opposite parity.

Project P' into the edge set of $\mathrm{HTG}(3, r, \ell)$ as follows. If $u_{0,x}u_{0,x+1}$ is an edge of P', where the subscripts are treated modulo r, then $u_{0,x}u_{0,x+1}$ is an edge of the projection in $\mathrm{HTG}(3, r, \ell)$. If $u_{0,x}u_{0,y}$ is a jump edge in P' with x odd and y even, then $u_{2,x}u_{0,y}$ is an edge in the projection in $\mathrm{HTG}(3, r, \ell)$. Let Y denote the subgraph of $\mathrm{HTG}(3, r, \ell)$ resulting from the projection of P'.

Let $u_{2,x_1}, u_{2,x_2}, \ldots, u_{2,x_t}$ be the end vertices in column 2 of the jump edges of Y, where $0 < x_1 < x_2 < \cdots < x_t < r$. Now add the flat edges $u_{0,x_a}u_{1,x_a}$ for $a = 1, 2, \ldots, t$. Vertical fills between columns 1 and 2 yield a Hamilton path from $u_{0,0}$ to $u_{0,j}$. Furthermore, if we also add the flat edge $u_{0,j}u_{1,j}$ and then do the vertical fills between columns 1 and 2, we obtain a Hamilton path from $u_{0,0}$ to $u_{2,j}$.

From the preceding, we see that whenever there is a Hamilton path from $u_{0,0}$ to $u_{0,j}$ in $\mathrm{HTG}(1, r, \ell)$ using at least one jump edge, then there are Hamilton paths from $u_{0,0}$ to both $u_{0,j}$ and $u_{2,j}$ in $\mathrm{HTG}(3, r, \ell)$. So the presence of jump edges is crucial.

A Hamilton path in $\mathrm{HTG}(1, r, \ell)$ from $u_{0,0}$ to $u_{0,j}$ must use jump edges if j is neither 1 nor $r - 1$. Because $u_{0,0}u_{0,1} \cdots u_{0,r-1}u_{0,0}$ is a Hamilton cycle in $\mathrm{HTG}(1, r, \ell)$, there is another Hamilton cycle C, by Smith's Theorem [18], using the edge $u_{0,0}u_{0,1}$. Clearly C must have at least one jump edge. The same argument applies to the edge $u_{0,0}u_{0,r-1}$. Therefore, for each $u_{0,j}$, j odd, there is a Hamilton path in $\mathrm{HTG}(3, r, \ell)$ from $u_{0,0}$ to both $u_{0,j}$ and $u_{2,j}$.

We now may obtain a Hamilton path from $u_{0,0}$ to any vertex of the form $u_{1,j}$, j even, because both of the following permutations are automorphisms of $\mathrm{HTG}(3, r, \ell)$:

- $f(u_{i,j}) = u_{i,j+2}$; and
- $g(u_{i,j}) = u_{i+1,j+1}$ for $i \in \{0, 1\}$ and $g(u_{2,j}) = u_{0,1+j+\ell}$.

Therefore, $\mathrm{HTG}(3, r, \ell)$ is Hamilton-laceable. □

5 Cycle Structure

We now look at cycles in honeycomb toroidal graphs with respect to two properties: girth and cycle spectrum. Throughout this section we use the important convention that the notation $\mathrm{HTG}(m, r, \ell)$ always is in *normal form*, that is, $\ell \le r/2$. This convention is possible because $\mathrm{HTG}(m, r, \ell)$ is isomorphic to $\mathrm{HTG}(m, r, r - \ell)$. Hence, the information given with respect to ℓ assumes $r \ge 2\ell$.

There are no odd length cycles because honeycomb toroidal graphs are bipartite. All $\mathrm{HTG}(m, r, \ell)$ contain 6-cycles (K_4 is not a honeycomb toroidal graph) implying that the girth is either 4 or 6. The next result handles the girth situation and is given without its easy proof.

Theorem 5.1. *The girth of* $\mathrm{HTG}(m, r, \ell)$ *is 6 with the following exceptions for which the girth is 4:*

- $r = 4$*;*
- $m = 1, r > 4$ *and* $\ell = 3$*;*
- $m = 1, r > 4, r \equiv 2 (\mathrm{mod}\ 4)$ *and* $\ell = r/2$*;*
- $m = 1, r > 4, r \equiv 0 (\mathrm{mod}\ 4)$ *and* $\ell = \frac{r-2}{2}$*; and*
- $m = 2, r > 4$ *and* $\ell \in \{0, 2\}$*.*

We now consider the cycle spectrum property. Recall that a graph is *even pancyclic* if it contains all possible even length cycles from length 4 through $2\lfloor N/2 \rfloor$, where N is the order of the graph. Given that connected bipartite

Cayley graphs of valency at least 3 on abelian groups are even pancyclic [1] and honeycomb toroidal graphs are Cayley graphs on groups that are close to being abelian, we expect that the latter graphs should have a rich cycle spectrum.

Cycles whose lengths are congruent to 2 modulo 4 have a straightforward answer according to the following result. We outline a proof with details to be filled in by the reader. Moreover, to help with the description, we now describe two operations that increase cycle lengths by 4 in honeycomb toroidal graphs.

Let C be a cycle in $\mathrm{HTG}(m,r,\ell)$ using the 2-path $u_{i,j}u_{i,j+1}u_{i+1,j+1}$, not passing through $u_{i,j+2}$ and not using any vertex of column $i-1$. We obtain a new cycle whose length has increased by 4 if we remove the edge $u_{i,j}u_{i,j+1}$ and replace it with the 5-path $u_{i,j+1}u_{i,j+2}u_{i-1,j+2}u_{i-1,j+1}u_{i-1,j}u_{i,j}$. We call this operation a *vertical bypass* of edge $u_{i,j}u_{i,j+1}$. We may think of this bypass as going up and to the left. It is clear that we may do the same going to the right and/or going down.

Now suppose we have a cycle C in $\mathrm{HTG}(1,r,\ell)$ using the edge $u_{0,j}u_{0,j+\ell}$ and not containing any of the vertices $\{u_{0,j-2},u_{0,j-1},u_{0,j+\ell-2},u_{0,j+\ell-1}\}$. Using the 5-path $u_{0,j}u_{0,j-1}u_{0,j-2}u_{0,j+\ell-2}u_{0,j+\ell-1}u_{0,j+\ell}$ to replace the edge $u_{0,j}u_{0,j+\ell}$ gives us a cycle whose length has increased by 4. We call this an *interval-shrinking bypass.*

Theorem 5.2. *The graph* $\mathrm{HTG}(m,r,\ell)$ *has cycles of length L for all L satisfying* $L \equiv 2 \pmod 4$ *and* $6 \le L \le mr$.

Proof. We have two cases based on the parity of m and first consider even m. It is easy to see how to obtain cycles of lengths 6 through $2r-2$ using vertices in columns 0 and 1, where the cycle of length $2r-2$ omits $u_{0,0}$ and $u_{1,0}$ and the cycle lengths are congruent to 2 modulo 4.

If $m=2$, we are finished. Otherwise, perform a vertical bypass to the right and down with the edge $u_{1,1}u_{1,2}$. This produces a cycle of length $2r+2$. Now do a vertical bypass to the right and up with the edge $u_{2,1}u_{2,2}$ to obtain a cycle of length $2r+6$. We then may produce cycles with lengths increasing by 4 at each iteration using columns 2 and 3 until reaching a cycle of length $4r-2$ omitting the vertices $u_{0,0}$ and $u_{3,0}$. We continue in this manner when m is even obtaining cycles of the required lengths until reaching $L=mr-2$ and completing this case.

We now move to m odd for which the key is $m = 1$ and that is where we begin. The strategy is to find sequences of successive lengths congruent to 2 modulo 4 so that the sequences possibly overlap and fill in all possible desired lengths from 6 through mr or $mr - 2$.

Start with the 6-cycle $u_{0,1}u_{0,\ell+1}u_{0,\ell+2}u_{0,\ell+3}u_{0,3}u_{0,2}u_{0,1}$. Perform interval-shrinking bypasses until reaching the cycle

$$u_{0,1}u_{0,2}\cdots u_{0,\ell-1}u_{0,2\ell-1}u_{0,2\ell-2}\cdots u_{0,\ell}u_{0,1}$$

of length 2ℓ. This gives us cycles of lengths $6, 10, 14, \ldots, 2\ell$ and this is the first of the aforementioned sequences.

The cycle $C_1 = u_{0,1}u_{0,2}u_{0,r-\ell+2}u_{0,r-\ell+1}u_{0,r-\ell}\cdots u_{0,\ell+1}u_{0,1}$ has length $r - 2\ell + 4$. Suppose that $r - 2\ell + 4 \equiv 2(\text{mod } 4)$ and note this implies r is a multiple of 4. Use interval-shrinking bypasses to get cycles of all lengths congruent to 2 modulo 4 from $r-2\ell+4$ through $r-2$ giving us the sequence of lengths $r - 2\ell + 4, r - 2\ell + 8, \ldots, r - 2$.

If $2\ell \geq r - 2\ell$, then we have cycles of all lengths congruent to 2 modulo 4. This happens when $\ell \geq r/4$. Otherwise, $\ell < r/4$.

When $\ell < r/4$, then the cycle

$$C_2 = u_{0,1}u_{0,\ell+1}u_{0,\ell}u_{0,2\ell}, u_{0,2\ell+1}u_{0,2\ell+2}\cdots u_{r-\ell+2}u_{0,2}u_{0,1}$$

has length $r - 3\ell + 7$. If $r - 3\ell + 7 \equiv 2(\text{mod } 4)$, then interval shrinking bypasses yield cycles of increasing lengths congruent to 2 modulo 4 until reaching length $r - \ell + 1$. This sequence of lengths overlaps the sequence beginning with $r - 2\ell + 4$ so that we now have all desired lengths from $r - 3\ell + 7$ to $r - 2$. If $2\ell \geq r - 3\ell + 3$, we are done. Otherwise, $\ell < (r + 3)/5$ and we continue in the same way by using a cycle starting with $u_{0,1}u_{0,\ell+1}u_{0,\ell}u_{0,2\ell}u_{0,2\ell-1}u_{0,3\ell-1}$. We maintain this pattern until we obtain all lengths congruent to 2 modulo 4.

The cycle we obtain by replacing the 4-path $u_{0,\ell+1}u_{0,\ell}u_{0,2\ell}u_{0,2\ell+1}u_{0,2\ell+2}$ of C_2 with the 2-path $u_{0,\ell+1}u_{0,\ell+2}u_{0,2\ell+2}$ gives us a cycle C_3 whose length is two less than the length of C_2. So if $r - 3\ell + 7 \not\equiv 2(\text{mod } 4)$, then we use C_3 instead of C_2 to obtain appropriate cycle lengths.

If $r - 2\ell + 4 \equiv 0(\text{mod } 4)$, the length of C_1, then replace the 2-path

$$u_{0,\ell+3}u_{0,\ell+4}u_{\ell+5}$$

of C_1 with

$$u_{0,\ell+3}u_{0,3}u_{0,4}u_{0,5}u_{0,\ell+5}$$

to obtain a cycle C_4 of length $r-2\ell+6$. We do as before to obtain longer cycles and the maximum length we reach is $r-4$, but this is sufficient as we have an r-cycle. The other lengths are obtained as in the preceding description and this takes care of $m=1$.

When $m>1$ is odd, we obtain cycles of all desired lengths from 6 through $(m-1)r-2$ using the technique for an even number of columns at the beginning of the proof on the last $m-1$ columns. We easily find a cycle of length $(m-1)r+2$ by using a vertical bypass. For lengths longer than $(m-1)r+2$, that is, length $(m-1)r+M$, $M\geq 6$, take a cycle of length M in $\mathrm{HTG}(1,r,\ell)$ and project it into $\mathrm{HTG}(m,r,\ell)$ as is done in the proof of Theorem 4.2. Then use vertical fills as in the proof of Theorem 4.1 to obtain a cycle of length $(m-1)r+M$ as required. □

From Theorems 5.1 and 5.2, we see that cycles whose lengths are multiples of 4 are of interest. Honeycomb toroidal graphs $\mathrm{HTG}(m,r,\ell)$ for $m\geq 3$ and $n>4$ have a simple 12-cycle lying in three columns so that lengths 4 and 8 become the only possible missing values once it is seen how to increase cycle lengths by 4 at a time. The cycle spectrum problem was settled for $\mathrm{HTG}(m,r,\ell)$, when $m\geq 3$, in [15]. We summarize their results in Table 1. Any of the graphs not listed in the table are even pancyclic. The missing even cycle lengths from 4 through $mn-2$ are displayed in the right column. We remind the reader that the entries in the table are in normal form.

Table 1

The graphs	Missing cycle lengths L
$\mathrm{HTG}(m,4,\ell)$, even $m\geq 6$	$L\equiv 0 \pmod 4$ and $4<L<2m$
$\mathrm{HTG}(m,4,\ell)$, odd $m\geq 5$	$L\equiv 0 \pmod 4$ and $4<L<2m+2$
$\mathrm{HTG}(m,r,\ell), m\geq 3, r=6,8$	$L=4$
$\mathrm{HTG}(3,r,\ell), r\geq 10$ $\ell\in\{1,3,5\}$	$L=4$
$\mathrm{HTG}(4,r,\ell), r\geq 10$ $\ell\in\{0,2,4\}$	$L=4$
$\mathrm{HTG}(4,r,\ell), r\geq 10$ $\ell\notin\{0,2,4\}$	$L=4,8$
$\mathrm{HTG}(m,r,\ell)$, even $m\geq 6, r\geq 10$	$L=4,8$
$\mathrm{HTG}(3,r,\ell), r\geq 10$ $\ell\notin\{1,3,5\}$	$L=4,8$
$\mathrm{HTG}(m,r,\ell)$, odd $m\geq 5, r\geq 10$	$L=4,8$

This leaves the spectrum problem unsettled for $m = 1$ and $m = 2$ when the cycle lengths are congruent to 0 modulo 4. The honeycomb toroidal graphs for $m = 1$ were seen to be crucial for the Hamilton laceability question so that they are an interesting subclass. As a side note, HTG$(1, 14, 5)$ is the Heawood graph so that the subclass contains well-known graphs. The information for these two values of m is contained in Table 2, but there are a couple of comments that are appropriate.

Table 2

The graphs	Missing cycle lengths L
HTG$(1, r, 3), r \geq 6$	none
HTG$(1, r, r/2), r \equiv 2(\text{mod } 4)$	none
HTG$(1, r, (r-2)/2), r \equiv 0(\text{mod } 4)$	none
HTG$(2, r, \ell), \ell \in \{0, 2\}$	none
HTG$(1, r, 5), r \geq 14$	$L = 4$
HTG$(1, r, 7), r > 14$	$L = 4$
HTG$(1, r, \ell), r \equiv 2(\text{mod } 4), r > 14$ odd $\ell \in \{(r-4)/2, (r-2)/4, (r+2)/4\}$	$L = 4$
HTG$(1, r, \ell), r \equiv 0(\text{mod } 4), r > 16$ odd $\ell \in \{(r-6)/2, (r-4)/4, r/4, (r+4)/4\}$	$L = 4$
HTG$(1, r, (r \pm 3)/3), r \equiv 0(\text{mod } 6), r > 18$	$L = 4$
HTG$(1, r, \ell), r \equiv 4(\text{mod } 6), r > 10$ $\ell \in \{(r-1)/3, (r+5)/3\}$	$L = 4$
HTG$(1, r, \ell), r \equiv 2(\text{mod } 6), r > 20$ $\ell \in \{(r-5)/3, (r+1)/3\}$	$L = 4$
HTG$(2, r, 4), r \geq 8$	$L = 4$
HTG$(2, r, (r-4)/2), r \equiv 0(\text{mod } 4), r > 8$	$L = 4$
HTG$(2, r, (r-2)/2), r \equiv 2(\text{mod } 4), r > 6$	$L = 4$

From Theorem 5.1, we know precisely which honeycomb toroidal graphs have no 4-cycles. On the other hand, when $r > 2\ell + 2$,

$$v_1 v_2 v_3 v_{\ell+3} v_{\ell+4} v_{2\ell+4} v_{2\ell+3} v_{2\ell+2} v_{2\ell+1} v_{2\ell} v_{\ell} v_{\ell+1} v_1,$$

where $v_j = u_{0,j}$, is a 12-cycle in HTG$(1, r, \ell)$ for $\ell > 5$. Lengths congruent to 0 modulo 4 longer than 12 are obtained using methods similar to those in the proof of Theorem 5.2.

For $m = 2$ it is straightforward to obtain a 12-cycle whenever $\ell \geq 4$ and the longer cycle lengths congruent to 0 modulo 4 are again obtained in a similar manner to what was done before. Hence, it 8-cycles that may be missing. For this reason, a convention for Table 2 that is different from Table 1 is that any $\mathrm{HTG}(1,r,\ell)$ or $\mathrm{HTG}(2,r,\ell)$ not mentioned in the table has neither 4-cycles nor 8-cycles, but all other even length cycles in the feasible range.

6 Paths And Diameter

The *diameter* of a connected graph is the maximum distance between pairs of distinct vertices in the graph. This parameter is of interest to anyone concerned with the propagation of information throughout a network. As this involves distances between vertices, we are interested in shortest paths in honeycomb toroidal graphs. The next two lemmas provide useful information about shortest paths in honeycomb toroidal graphs. Some terminology is necessary before stating them.

When talking about directions in which edges are traversed, travelling along a flat edge from column i to column $i+1$ is one direction and travelling from column $i+1$ to column i is the other direction. Similarly, the two directions for jump edges are from column 0 to column $m-1$ and vice versa.

Lemma 6.1. *Every jump edge in a shortest path in* $\mathrm{HTG}(m,r,\ell)$ *is traversed in the same direction.*

Proof. If a shortest path contains no jump edge, there is nothing to prove so let P be a shortest path in $\mathrm{HTG}(m,r,\ell)$ containing a jump edge. Suppose the first jump edge encountered when traversing P is $u_{0,j}u_{m-1,j-\ell}$, that is, we traverse it from column 0 to column $m-1$. Suppose the next jump edge encountered along P has the form $u_{m-1,k}u_{0,k+\ell}$, that is, it is traversed from column $m-1$ to column 0.

This implies that the subpath P' of P from $u_{m-1,j-\ell}$ to $u_{m-1,k}$ has no jump edges and the second subscript has changed from $j-\ell$ to k. This is done only by vertical edges in various columns. The change from j to $k+\ell$ is the same as the change from $j-\ell$ to k. Hence, we may delete the subpath from $u_{0,j}$ to $u_{0,k+\ell}$ and replace it with the vertical changes in P' translated by ℓ projected onto column 0 . This gives us a shorter walk (some edges may be duplicated via the projection) from $u_{0,0}$ to the terminal vertex of P. This is a contradiction to P being a shortest path.

A similar contradiction arises if the traversals of two consecutive jump edges are reversed. Therefore, all the jump edges in a shortest path go from column 0 to column $m-1$ or vice versa. $\square$

Lemma 6.2. *Let a shortest path P in* $\mathrm{HTG}(m,r,\ell)$ *have jump edges. If P contains flat edges between the same two columns, they must be separated by a jump edge. In particular, if P has no jump edges, then there is at most one flat edge between two columns in P.*

Proof. Let $u_{i,j}u_{i+1,j}$ and $u_{i,k}u_{i+1,k}$ be succesive appearances of flat edges between columns i and $i+1$, $0 \le i < m-1$. Suppose that the first edge is traversed from $u_{i,j}$ to $u_{i+1,j}$. If there is no jump edge between $u_{i,j}u_{i+1,j}$ and the edge $u_{i,k}u_{i+1,k}$, then there are vertical edges taking the second subscript from j to k no matter which direction $u_{i,k}u_{i+1,k}$ is traversed.

In either case, remove the subpath of P from $u_{i,j}$ to $u_{i,k}$ and replace it with the projection of the vertical edges onto column i. This yields a shorter walk with the same terminal vertices which is a contradiction. Similar arguments work if the edge $u_{i,j}u_{i+1,j}$ is traversed in the opposite direction. The conclusion follows from this. $\square$

Consider the special graph $\mathrm{HTG}(m,r,0)$. If we are looking for a shortest path from $u_{0,0}$ to $u_{i,j}$, it is clear that we need vertical edges taking us to row j and flat edges (the jump edge is also flat in this case) taking us to column i. So if $i \le m/2$, we use flat edges in the direction left to right, and if $i > m/2$, we take a jump edge from column 0 to column $m-1$ followed by flat edges from right to left. We use vertical edges as required to reach row j. It is straightforward to obtain the diameter as shown in Table 3.

The preceding worked easily because the jump edges change the second subscript by zero. Other values for the jump edges allow for big changes in shortest paths because a large jump edge value allows large changes in the second subscript. For example, suppose we are trying to increase the second subscript as much as possible. We can start a path at $u_{0,0}$ and reach the vertex $u_{m-1,m-1}$ when we first reach column $m-1$. We follow this with the edge $u_{m-1,m-1}u_{m-1,m}$ and then the jump edge $u_{m-1,m}u_{0,m+\ell}$.

We now have a path from $u_{0,0}$ to $u_{0,m+\ell}$ of length $2m$. If instead we took the path from $u_{0,0}$ to $u_{0,m+\ell}$ up column 0, it has length $m+\ell$. Thus, if $\ell > m$, we have a shorter path by using a jump edge. Lemma 6.1 provides some help because it tells us that if we use more than one jump edge in a shortest path, we must use them in the same direction which forces many edges to be used between their appearances.

Research Problem 2. Determine the shortest paths between vertices in an arbitrary $\mathrm{HTG}(m, r, \ell)$.

The diameters of a few honeycomb toroidal graphs have been determined in [17, 19, 21] and we summarize their results in the following table. Note that [19] corrects an error for the diameter of $\mathrm{HTG}(m, 2m, m)$ given in [17].

Table 3

The graphs	diameter
$\mathrm{HTG}(m, 6m, 3m)$	$2m$
$\mathrm{HTG}(m, 2m, m), m \geq 2, m \equiv 1, 4 (\mathrm{mod}\ 6)$	$\lfloor 4m/3 \rfloor$
$\mathrm{HTG}(m, 2m, m), m \geq 2, m \equiv 0, 2, 3, 5 (\mathrm{mod}\ 6)$	$\lceil 4m/3 \rceil$
$\mathrm{HTG}(m, r, 0), m$ even $, m \geq r - 2$	m
$\mathrm{HTG}(m, r, 0), m$ even $, m < r - 2$	$(r + m)/2$
$\mathrm{HTG}(m, r, \ell), m \geq r/2, \ell \equiv r - m (\mathrm{mod}\ r)$	$\max\{m, \lfloor (2m + r + 1)/3 \rfloor$

Research Problem 3. Determine the diameter of $\mathrm{HTG}(m, r, \ell)$ in terms of the parameters m, r and ℓ.

The preceding problem undoubtedly has many subcases as the value of the jump varies. Lemmas 6.1 and 6.2 allow us to determine that the diameter of $\mathrm{HTG}(1, r, \ell)$ is $2\lfloor r/\ell \rfloor + 1$ whenever $\ell \leq \sqrt{r}$. We shall not present the tedious proof of this fact, but mention it just to indicate the kinds of complications that likely arise in considering the preceding problem.

7 Automorphisms

Honeycomb toroidal graphs are Cayley graphs [4] on a generalized dihedral group. This means they are vertex-transitive. As mentioned earlier, $\mathrm{HTG}(1, 14, 5)$ is the Heawood graph and its automorphism group has order

336 in spite of the graph having only 14 vertices. On the other hand, the automorphism group of $\mathrm{HTG}(1,14,3)$ has order only 28. So we see there may be wide variations in the automorphism groups of these graphs. This suggests the next problem.

Research Problem 4. Determine the automorphism group of an arbitrary $\mathrm{HTG}(m,r,\ell)$ in terms of the parameters m,r and ℓ.

Given a family of Cayley graphs, there is interest in determining those with minimal automorphism groups. In this case that means those that are GRRs, that is, those for which $|\mathrm{Aut}(\mathrm{HTG}(m,r,\ell)| = mr$.

Research Problem 5. Determine when $\mathrm{HTG}(m,r,\ell)$ is a GRR, that is, $|\mathrm{Aut}(\mathrm{HTG}(m,r,\ell))| = mr$.

Little is known about the preceding question. One result in this direction comes from [10] in which the following result is proved.

Theorem 7.1. *The graph* $\mathrm{HTG}(1,r,\ell)$ *in normal form is a* GRR *if and only if* $r \geq 18, \ell < r/2$ *and the following all hold:*

- $(\ell+1)^2/4 \not\equiv 1 (\mathrm{mod}\ r/2)$*;*
- $(\ell-1)^2/4 \not\equiv 1 (\mathrm{mod}\ r/2)$*; and*
- $(\ell^2-1)/4 \not\equiv -1 (\mathrm{mod}\ r/2)$*.*

8 Conclusion

The family of graphs under discussion is of interest for several reasons and we have looked at it primarily from a graph theoretic viewpoint. There has been considerable work done on algorithmic aspects of honeycomb toroidal graphs. Most of the concern is with routing, broadcasting, bisection width, semigroup computation and cost. Again, most of the work has dealt with the special honeycomb toroidal graphs introduced in [17] and their extensions. So there is room for research for the entire family of honeycomb toroidal graphs. There is background for the algorithmic work in [6, 8, 11, 14, 16, 17].

We also have presented some specific research problems that we find interesting. This is a family of graphs worthy of much further investigation.

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