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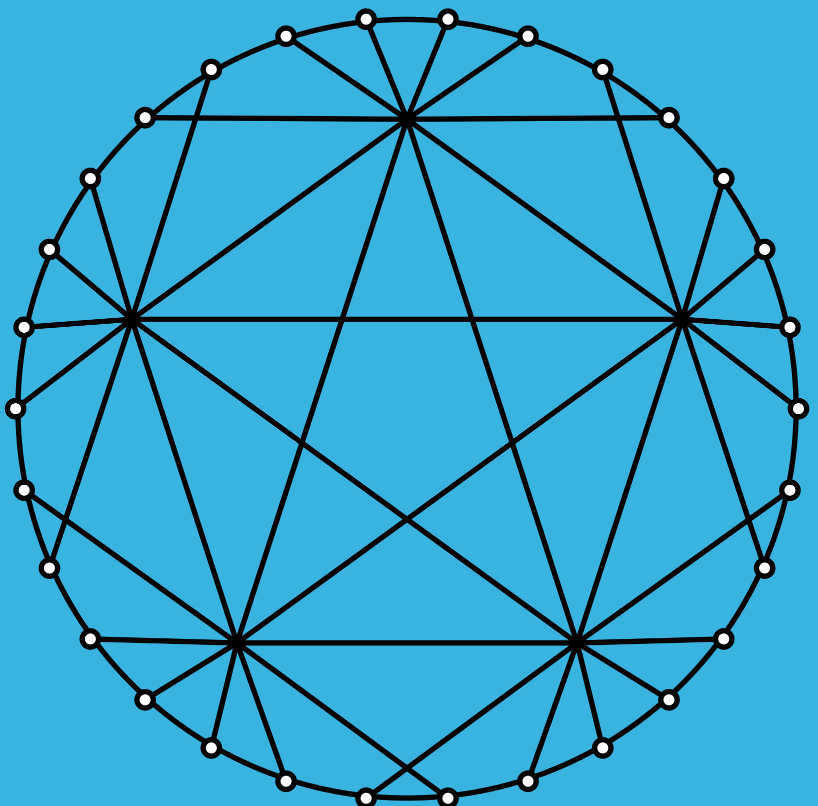

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Designing progressive dinner parties

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Abstract: I recently came across a combinatorial design problem involving progressive dinner parties (also known as safari suppers). In this note, I provide some elementary methods of designing schedules for these kinds of dinner parties.

1 The problem

A simple form of *progressive dinner party* could involve three couples eating a three-course dinner, with each couple hosting one course. I received email from Julian Regan asking if there was a nice way to design a more complicated type of progressive dinner party, which he described as follows:

> The event involves a number of couples having each course of a three-course meal at a different person's house, with three couples at each course, every couple hosting once and no two couples meeting more than once.

Let us represent each couple by a *point* $x \in X$ and each course of each meal by a *block* consisting of three points. Suppose there are v points (i.e., couples). Evidently we want a collection of blocks of size three, say $\mathcal{B}$, such that the following conditions are satisfied:

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1. The blocks can be partitioned into three parallel classes, each consisting of $v/3$ disjoint blocks. (Each parallel class corresponds to a specific course of the meal.) Hence, there are a total of v blocks and we require $v \equiv 0 \bmod 3$.
2. No pair of points occurs in more than one block.
3. There is a bijection $h : \mathcal{B} \to X$ such that $h(B) \in B$ for all $B \in \mathcal{B}$. (That is, we can identify a *host* for each block in such a way that each point occurs as a host exactly once.)

We will refer to such a collection of blocks as a PDP(v).

It is not hard to see that a PDP(v) does not exist if $v = 3$ or $v = 6$, because we cannot satisfy condition 2. However, for all larger values of v divisible by three, we show in Section 2 that it is possible to construct a PDP(v). Section 3 considers a generalization of the problem in which there are k courses and k couples present at each course, and gives a complete solution when $k = 4$ or $k = 5$.

2 Two solutions

We begin with a simple construction based on latin squares. A *latin square* of order n is an n by n array of n symbols, such that each symbol occurs in exactly one cell in each row and each column of the array. A *transversal* of a latin square of order n is a set of n cells, one from each row and each column, that contain n different symbols. Two transversals are *disjoint* if they do not contain any common cells.

Lemma 2.1. *Suppose there is a latin square of order w that contains three disjoint transversals. Then there is a PDP($3w$).*

Proof. Let L be a latin square of order w that contains disjoint transversals T_1, T_2 and T_3. Let the rows of L be indexed by R, let the columns be indexed by C and let the symbols be indexed by S. We assume that R, C and S are three mutually disjoint sets. Each transversal T_i consists of w ordered pairs in $R \times C$.

We will construct a PDP($3w$) on points $X = R \cup C \cup S$. For $1 \leq i \leq 3$, we construct a parallel class P_i as follows:

$$P_i = \{\{r, c, L(r, c)\} : (r, c) \in T_i\}.$$

Finally, for any block $B = \{r, c, s\} \in P_1 \cup P_2 \cup P_3$, we define $h(B)$ as follows:

- if $B \in P_1$, then $h(B) = r$
- if $B \in P_2$, then $h(B) = c$
- if $B \in P_3$, then $h(B) = s$.

The verifications are straightforward.

- First, because each T_i is a transversal, it is clear that each P_i is a parallel class.
- No pair of points $\{r, c\}$ occurs in more than one block because the three transversals are disjoint.
- Suppose a pair of points $\{r, s\}$ occurs in more than one block. Then there is $L(r, c) \in T_i$ and $L(r, c') \in T_j$ such that $L(r, c) = L(r, c')$. T_i and T_j are disjoint, so $c \neq c'$. But then we have two occurrences of the same symbol in row r of L, which contradicts the assumption that L is a latin square.
- The argument that no pair of points $\{c, s\}$ occurs in more than one block is similar.
- Finally, the mapping h satisfies property 3 because each T_i is a transversal. □

Theorem 2.2. *There is a PDP$(3w)$ for all $w \geq 3$.*

Proof. If ≥ 3, $w \neq 6$, there is a pair of orthogonal latin squares of order w. It is well-known that a pair of orthogonal latin squares of order w is equivalent to a latin square of order w that contains w disjoint transversals (see, e.g., [3, p. 162]). Since $w \geq 3$, we have three disjoint transversals and we can apply Lemma 2.1 to obtain a PDP(w). There do not exist a pair of orthogonal latin squares of order 6, but there is a latin square of order 6 that contains four disjoint transversals (see, e.g., [3, p. 193]). So we can also use Lemma 2.1 to construct a PDP(18). □

Example 2.1. *We construct a PDP*(12)*. Start with a pair of orthogonal latin squares of order* 4*:*

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 2 \\ \hline 4 & 2 & 1 & 3 \\ \hline 2 & 4 & 3 & 1 \\ \hline 3 & 1 & 2 & 4 \\ \hline \end{array}, \quad L_2 = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 2 & 3 \\ \hline 3 & 2 & 4 & 1 \\ \hline 4 & 1 & 3 & 2 \\ \hline 2 & 3 & 1 & 4 \\ \hline \end{array}.$$

Each symbol in L_2 *gives us a transversal in* L_1*. Suppose we index the rows by* r_i *(*$1 \leq i \leq 4$*) and the columns by* c_j *(*$1 \leq j \leq 4$*). From symbols* $1, 2$ *and* 3*, we obtain the following three disjoint transversals in* L_1*:*

$$\begin{aligned} T_1 &= \{(r_1, c_1), (r_2, c_4), (r_3, c_2), (r_4, c_3)\} \\ T_2 &= \{(r_1, c_3), (r_2, c_2), (r_3, c_4), (r_4, c_1)\} \\ T_3 &= \{(r_1, c_4), (r_2, c_1), (r_3, c_3), (r_4, c_2)\}. \end{aligned}$$

Suppose we relabel the points as $1, \dots 12$*, replacing* $r_1, \dots, r_4$ *by* $1, \dots, 4$*; replacing* $c_1, \dots, c_4$ *by* $5, \dots, 8$*; and replacing the symbols* $1, \dots, 4$ *by* $9, \dots, 12$*. Then we obtain the following PDP*(12)*, where the hosts are indicated in red:*

$$\begin{aligned} P_1 &= \{\{1, 5, 9\}, \{2, 8, 11\}, \{3, 6, 12\}, \{4, 7, 10\}\} \\ P_2 &= \{\{1, 7, 12\}, \{2, 6, 10\}, \{3, 8, 9\}, \{4, 5, 11\}\} \\ P_3 &= \{\{1, 8, 10\}, \{2, 5, 12\}, \{3, 7, 11\}, \{4, 6, 9\}\}. \end{aligned}$$

Of course, using a pair of latin squares is overkill. It would perhaps be easier just to give explicit formulas to construct a PDP. Here is one simple solution that works for all $v \geq 9$ such that $v \equiv 0 \bmod 3$ and $v \neq 12$.

Theorem 2.3. *Let* $w \geq 3$*,* $w \neq 4$*, and let* $X = \mathbb{Z}_w \times \{0, 1, 2\}$*. Define the following three parallel classes:*

$$\begin{aligned} P_0 &= \{\{(0,0), (0,1), (0,2)\} \bmod w\} \\ P_1 &= \{\{(0,0), (1,1), (2,2)\} \bmod w\} \\ P_2 &= \{\{(0,0), (2,1), (4,2)\} \bmod w\}. \end{aligned}$$

For any block $B = \{(i,0), (j,1), (k,2)\} \in P_0 \cup P_1 \cup P_2$*, define* $h(B)$ *as follows.*

- *if* $B \in P_0$*, then* $h(B) = (i, 0)$
- *if* $B \in P_1$*, then* $h(B) = (j, 1)$
- *if* $B \in P_2$*, then* $h(B) = (k, 2)$*.*

Then P_0, P_1, P_2*, and* h *yield a PDP*$(3w)$*.*

Proof. It is clear that each P_i is a parallel class because we are developing a base block modulo w and each base block contains one point with each possible second coordinate. For the same reason, the mapping h satisfies property 3.

Consider the differences $(y-x) \bmod w$ that occur between pairs of points $\{(x,0),(y,1)\}$. We obtain all pairs with differences $0,1$ and 2 when we develop the three base blocks. The same thing happens when we look at the differences $(y-x) \bmod w$ between pairs of points $\{(x,1),(y,2)\}$.

Finally, consider the differences $(y-x) \bmod w$ that occur between pairs of points $\{(x,0),(y,2)\}$. We obtain all pairs with differences $0,2$ and 4 modulo w when we develop the three base blocks. Since $w \neq 4$, these differences are distinct and the pairs obtained by developing the base blocks are also distinct. □

If $w = 4$, then the construction given in Theorem 2.3 does not yield a PDP(12), because various pairs occur in more than one block. For example, the pair $\{(0,0),(0,2)\}$ occurs in a block of P_0 as well as in a block of P_2.

Example 2.2. *We apply Theorem 2.3 with $w = 5$. The three parallel classes, with hosts in red, are:*

P_0	P_1	P_2
$\{(0,0),(0,1),(0,2)\}$	$\{(0,0),(1,1),(2,2)\}$	$\{(0,0),(2,1),(4,2)\}$
$\{(1,0),(1,1),(1,2)\}$	$\{(1,0),(2,1),(3,2)\}$	$\{(1,0),(3,1),(0,2)\}$
$\{(2,0),(2,1),(2,2)\}$	$\{(2,0),(3,1),(4,2)\}$	$\{(2,0),(4,1),(1,2)\}$
$\{(3,0),(3,1),(3,2)\}$	$\{(3,0),(4,1),(0,2)\}$	$\{(3,0),(0,1),(2,2)\}$
$\{(4,0),(4,1),(4,2)\}$	$\{(4,0),(0,1),(1,2)\}$	$\{(4,0),(1,1),(3,2)\}$

2.1 Finding hosts

The specific constructions that we provided in Section 2 led to a very simple method to identify hosts. However, no matter what collection of three parallel classes we use, it will be possible to define hosts in such a way that property 3 of a PDP will be satisfied.

Theorem 2.4. *Suppose that P_1, P_2 and P_3 are three parallel classes of blocks of size three, containing points from a set X of size $v \equiv 0 \bmod 3$. Then we can define a mapping h that satisfies property 3.*

Proof. Construct the bipartite point-block incidence graph of the design. The nodes in this graph are all the elements of $X \cup \mathcal{B}$. For $x \in X$ and $B \in \mathcal{B}$, we create an edge from x to B if and only if $x \in B$. The resulting graph is a 3-regular bipartite graph and hence it has a perfect matching M (this is a corollary of Hall's Theorem, e.g., see [2, Corollary 16.6]). For every $B \in \mathcal{B}$, define $h(B) = x$, where x is the point matched with B in the matching M. □

The following corollary is immediate.

Corollary 2.5. *Suppose that P_1, P_2 and P_3 are three parallel classes of blocks of size three, containing points from a set X of size $v \equiv 0 \bmod 3$. Suppose also that no pair of points occurs in more one block in $\mathcal{B} = P_1 \cup P_2 \cup P_3$. Then there is a PDP$(v)$.*

3 A generalization

Suppose we now consider a generalization where meals have k courses and each course includes k couples. We define a PDP(k, v) to be a set of blocks of size k, defined on a set of v points, which satisfies the following properties:

1. The blocks can be partitioned into k parallel classes, each consisting of v/k disjoint blocks. Hence, there are a total of v blocks and we require $v \equiv 0 \bmod k$.
2. No pair of points occurs in more than one block.
3. There is a bijection $h : \mathcal{B} \to X$ such that $h(B) \in B$ for all $B \in \mathcal{B}$.

The problem we considered in Section 1 was just the special case $k = 3$ of this general definition.

Here is a simple necessary condition for existence of a PDP(k, v).

Lemma 3.1. *If a PDP(k, v) exists, then $v \geq k^2$.*

Proof. A given point x occurs in k blocks, each having size k. The points in these blocks (excluding x) must be distinct. Therefore,

$$v \geq k(k-1) + 1 = k^2 - (k-1).$$

Since k divides v, we must have $v \geq k^2$. □

We have the following results that are straightforward generalizations of our results from Section 2. The first three of these results are stated without proof.

Lemma 3.2. *Suppose there are $k-2$ orthogonal latin squares of order w that contain k disjoint common transversals. Then there is a PDP(k, kw).*

Corollary 3.3. *Suppose there are $k-1$ orthogonal latin squares of order w. Then there is a PDP(k, kw).*

Theorem 3.4. *Suppose that $P_1, \ldots, P_k$ are k parallel classes of blocks of size k, containing points from a set X of size $v \equiv 0 \bmod k$. Then we can define a mapping h that satisfies property 3.*

Our last construction generalizes Theorem 2.3.

Theorem 3.5. *Let $w \geq k \geq 3$. Suppose that the following condition holds:*

$$\text{There is no factorization } w = st \text{ with } 2 \leq s \leq k-1 \text{ and } 2 \leq t \leq k-1. \tag{1}$$

Then there is a PDP(k, kw).

Proof. Define $X = \mathbb{Z}_w \times \{0, \ldots, k-1\}$ and define the following k parallel classes, $P_0, \ldots, P_{k-1}$:

$$P_i = \{\{(0,0),(i,1),(2i,2),\ldots,((k-1)i,k-1)\} \bmod w\},$$

for $i = 0, \ldots, k-1$. Finally, define the mapping h as follows. For any block $B \in P_\ell$, define $h(B) = (x, \ell)$, where (x, ℓ) is the unique point in B having second coordinate equal to ℓ. Then $P_0, \ldots, P_{k-1}$ and h yield a PDP(k, kw).

Most of the verifications are straightforward, but it would perhaps be useful to see how condition (1) arises. Consider the differences $(y-x) \bmod w$ that occur between pairs of points $\{(x,c),(y,c+d)\}$, where c and d are fixed, $0 \leq c \leq k-2$, $1 \leq d \leq k-c-1$. These difference are

$$0, d, 2d, \ldots, (k-1)d \bmod w,$$

where $0 < d \leq k-1$. We want all of these differences to be distinct. Suppose that

$$id \equiv jd \bmod w$$

where $0 \leq j < i \leq k-1$. Then

$$(i-j)d \equiv 0 \bmod w.$$

Hence,

$$ed \equiv 0 \bmod w$$

where $0 < e \leq k-1$ and $0 < d \leq k-1$. Then, it not hard to see that w can be factored as the product of two positive integers, both of which are at most $k-1$.

Conversely, suppose such a factorization exists, say $w = st$. Then the pair $\{(0,0),(0,t)\}$ occurs in a block in P_0 and again in a block in P_s. □

Observe that condition (1) of Theorem 3.5 holds if w is prime or if $w > (k-1)^2$. Therefore we have the following corollary of Theorem 3.5.

Corollary 3.6. *Let $w \geq k \geq 3$. Suppose that w is prime or $w > (k-1)^2$. Then there is a PDP(k, kw).*

In general, some values of w will be ruled out (in the sense that Theorem 3.5 cannot be applied) for a given value of k. For example, as we have already seen in the previous section, we cannot take $w = 4$ in Theorem 3.5 if $k = 3$. However, a PDP(12) was constructed by a different method in Example 2.1.

We have the following complete results for $k = 4$ and $k = 5$.

Theorem 3.7. *There is a PDP$(4, 4w)$ if and only if $w \geq 4$. Further, there is a PDP$(5, 5w)$ if and only if $w \geq 5$.*

Proof. For $k = 4$, we proceed as follows. Theorem 3.5 yields a PDP$(4, 4w)$ for all $w \geq 4$, $w \neq 4, 6, 9$. Theorem 3.3 provides a PDP$(4, 16)$ and a PDP$(4, 36)$ since three orthogonal latin squares of orders 4 and 9 are known to exist (see [3]). The last case to consider is $w = 6$. Here we can use a resolvable 4-GDD of type 3^8 ([4]). Actually, we only need four of the seven parallel classes in this design. Then, to define the hosts, we can use Theorem 3.4.

We handle $k = 5$ in a similar manner. Theorem 3.5 yields a PDP$(5, 5w)$ for all $w \geq 5$, $w \neq 6, 8, 9, 12$ or 16. There are four orthogonal latin squares of orders $8, 9, 12$ and 16 (see [3]) so these values of w are taken care of by Theorem 3.3.

Finally, the value $w = 6$ is handled by a direct construction due to Marco Buratti [1]. Define $X = \mathbb{Z}_{30}$ and

$$\mathcal{B} = \{\{0, 1, 8, 12, 14\} \bmod 30\}.$$

So we have thirty blocks that are obtained from the base block $B_0 = \{0, 1, 8, 12, 14\}$. It is easy to check that no pair of points is repeated, because the differences of pairs of points occurring in B_0 are all those in the set

$$\pm\{1, 2, 4, 6, 7, 8, 11, 12, 13, 14\}.$$

Define

$$P_0 = \{B_0 + 5j \bmod 30 : j = 0, 1, \ldots, 5\}$$

and for $1 \leq i \leq 4$, let

$$P_i = \{B + i \bmod 30 : B \in P_0\}.$$

In this way, $\mathcal{B}$ is partitioned into five parallel classes, each containing six blocks.

Theorem 3.4 guarantees that we can define hosts in a suitable fashion. However, it is easy to write down an explicit formula, namely, $h(B_0 + i) = i$ for $0 \leq i \leq 29$. □

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