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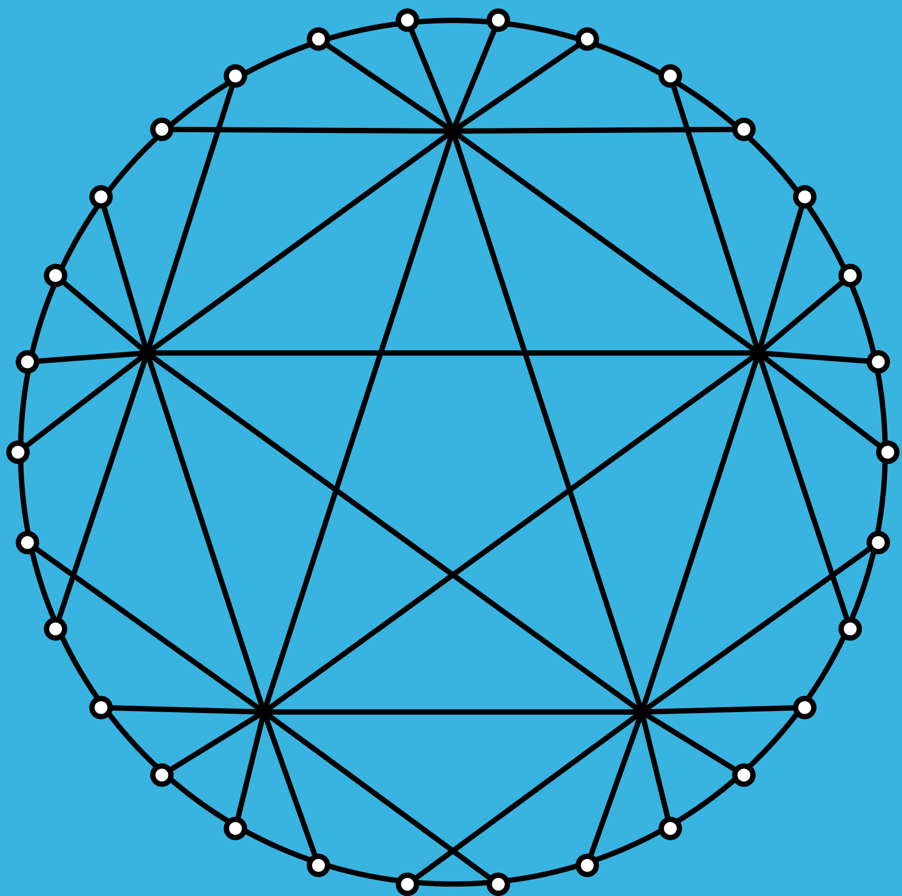

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Nonsequenceable Steiner triple systems

Donald L. Kreher*[1] and Douglas R. Stinson[2]

[1]Michigan Technological University, Houghton, Michigan, U.S.A.
kreher@mtu.edu

[2]University of Waterloo, Waterloo, Ontario, Canada
dstinson@uwaterloo.ca

Abstract: A partial Steiner triple system is is *sequenceable* if the points can be sequenced so that no proper segment can be partitioned into blocks. We show that, if $0 \le a \le (n-1)/3$, then there exists a nonsequenceable PSTS (n) of size $\frac{1}{3}\binom{n}{2} - a$, for all $n \equiv 1 \pmod 6$ except for $n = 7$.

1 Introduction

A decomposition of the complete graph on n points into triangles is called a *Steiner triple system* of order n and is denoted by STS (n). The vertex sets of the triples used are called the *blocks* of the Steiner triple system. Thus, equivalently, an STS (n) is a pair (X, B), where X is an n-element set of *points* and B is a collection of 3-element subsets of X called *blocks*, such that every pair of points is contained in exactly one block. It is well known that an STS (n) exists if and only if $n \equiv 1, 3 \pmod 6$. A decomposition of a proper subgraph of the complete graph on n points into triangles is called a *partial Steiner triple system* of order n and is denoted by PSTS (n). The *size* of a PSTS (n) is the number of blocks it contains.

An STS (n) is *sequenceable* if the points can be sequenced so that no proper segment can be partitioned into blocks. Such a sequence is called an *admissible sequence*. If an STS (n) has no admissible sequence, then we say it is

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nonsequenceable. For example, $\{abd, bce, cdf, deg, efa, fgb, gac\}$, the unique (up to isomorphism) STS (7) has the admissible sequence *abcdefg*. A fascinating study of sequenceable partial Steiner triple systems can be found in [1].

In [3] a related problem is examined. In this article the authors ask if the vertices of a Steiner triple system can be sequenced such that no length ℓ segment of the sequence contains a block. They obtain results for $\ell = 3$ and 4.

A set of disjoint blocks in an STS (n) is called a *partial parallel class.* Clearly any partial parallel class contains at most $\lfloor v/3 \rfloor$ blocks. A partial parallel class containing all the points of a design is called a *parallel class* and a partial parallel class containing all but one of the points of a design is called an *almost parallel class.*

Theorem 1.1. *Suppose a* PSTS (n) *has the property, for* $n-1$ *distinct points* x*, that there is an almost parallel class that does not contain* x*. Then the* PSTS (n) *is nonsequenceable.*

Proof. Consider any sequence $x_1x_2\cdots x_n$ of the vertices of such a PSTS (n). There is either an almost parallel class that does not contain x_1 or one that does not contain x_n. Thus the blocks of the first almost parallel class, if it exists, would partition the segment $x_2\cdots x_{n-1}x_n$, and the latter would partition the segment $x_1x_2\cdots x_{n-1}$. □

2 Example applications of Theorem 1.1

STS(13) : Developing the base blocks

$\{0,2,7\}$, $\{0,1,4\}$,

modulo 13 generates an STS (13) that contains the almost parallel class

$\{7,8,11\}$, $\{3,5,10\}$, $\{2,4,9\}$, $\{1,6,12\}$.

Because any translate of this almost parallel class is again an almost parallel class, there is an almost parallel class missing any desired point. Thus, by Theorem 1.1 this STS (13) is nonsequenceable.

STS(19) : Developing the base blocks

$\{0,1,6\}$, $\{0,2,10\}$, $\{0,3,7\}$,

modulo 19 generates an STS (19) that contains the almost parallel class

$$\{1,3,11\},\ \{2,15,16\},\ \{4,17,18\},\ \{5,8,12\},\ \{6,9,13\},\ \{7,10,14\}.$$

Thus by Theorem 1.1 this STS (19) is nonsequenceable.

STS(25) : Developing the base blocks

$$\big\{(0,0),(1,1),(0,2)\big\},\ \big\{(0,0),(0,1),(2,3)\big\},\ \big\{(0,0),(1,2),(2,0)\big\},\ \big\{(0,0),(1,0),(3,1)\big\}$$

modulo 5 independently in both coordinates generates a STS (25) with vertices $\mathbb{Z}_5 \times \mathbb{Z}_5$ that contains the almost parallel class

$$\big\{(0,1),(0,2),(2,4)\big\},\ \big\{(1,0),(3,2),(1,4)\big\},\ \big\{(1,1),(4,1),(0,3)\big\},\ \big\{(2,0),(2,3),(3,4)\big\},\ \big\{(2,1),(2,2),(4,4)\big\},\ \big\{(3,0),(1,2),(1,3)\big\},\ \big\{(3,1),(4,2),(3,3)\big\},\ \big\{(4,0),(4,3),(0,4)\big\}.$$

It follows from Theorem 1.1 that this STS (25) is nonsequenceable.

STS(31) : Developing the base blocks

$$\{0,5,11\},\ \{0,4,12\},\ \{0,3,13\},\ \{0,2,9\},\ \{0,1,15\},$$

modulo 31 generates an STS (31) that contains the almost parallel class

$$\{11,15,23\},\ \{10,26,27\},\ \{9,14,20\},\ \{8,13,19\},\ \{7,25,28\},\ \{5,21,22\},\ \{4,24,29\},\ \{3,6,16\},\ \{2,12,30\},\ \{1,17,18\}.$$

Thus by Theorem 1.1 this STS (31) is nonsequenceable.

STS(43) : Developing the base blocks

$$\{0,1,16\},\ \{0,2,14\},\ \{0,3,11\},\ \{0,4,37\},\ \{0,5,25\},\ \{0,7,24\},\ \{0,9,22\},$$

modulo 43 generates an STS (43) that contains the almost parallel class

$$\{14,15,30\},\ \{1,28,29\},\ \{2,20,25\},\ \{3,23,41\},\ \{4,33,35\},\ \{5,34,36\},\ \{6,19,40\},\ \{7,39,42\},\ \{8,26,31\},\ \{9,27,32\},\ \{10,37,38\},\ \{11,17,21\},\ \{12,18,22\},\ \{13,16,24\}.$$

Thus by Theorem 1.1 this STS (43) is nonsequenceable.

3 Constructions

A group divisible design with blocks of size 3, having u groups of size g and v groups of size m (denoted by 3-GDD of type $g^u m^v$), is a decomposition of the complete multipartite graph

$$K_{\underbrace{g,g,\ldots,g}_{u},\underbrace{m,m,\ldots,m}_{v}}$$

into triangles. The triangles (or triples) used in the decomposition are the blocks of the GDD and the partite sets are the groups. If $6 \mid m$ and $6 \mid g$, then a 3-GDD of type m^u and a 3-GDD of type $g^u m^1$ exist for all $u \geq 3$ [2, 4].

Theorem 3.1. *There exists a nonsequenceable* STS (n) *for all* $n \equiv 1 \pmod 6$ *except for* $n = 7$.

Proof. First suppose $n = 1 + 6k$ with $k = 2u$. If $u = 1$, then $n = 13$ and if $u = 2$, then $n = 25$. For both orders $n = 13$ or 25, there is an example of a nonsequenceable STS (n) given in Section 2.

If $u \geq 3$, there exist a 3-GDD of type 12^u with groups $G_1, G_2, \ldots, G_u$. Let X be a new point. In each group G_i, fill in $G_i \cup \{X\}$ with the nonsequenceable STS (13) found in Section 2.

We now show this STS satisfies the conditions of Theorem 1.1. Clearly it is satisfied for the point X. For any other point, say $y \in G_i$, take the almost parallel class in the i-th STS (13) that misses y, and for all $j \neq i$, take the almost parallel class in the j-th STS (13) that misses X. The union of these partial parallel classes is an almost parallel class that misses y.

Now suppose $n = 1 + 6k$ with $k = 2u + 3$. If $u = -1$, then $n = 7$ and, as noted in the Theorem, a nonsequenceable STS (7) does not exist. If $u = 0, 1$ or 2, then $n = 19, 31$ or 43 respectively. There is an example of a nonsequenceable STS (n) given in of Section 2 for each $n \in \{19, 31, 43\}$. If $u \geq 3$, there exists a 3-GDD of type $12^u 18^1$ with groups (partite sets) $G_1, G_2, \ldots, G_u, M$, where $|G_i| = 12$ for all i and $|M| = 18$. Let X be a new point. In each group G_i, fill in $G_i \cup \{X\}$ with the nonsequenceable STS (13) given in Section 2. On $M \cup \{X\}$, place the nonsequenceable STS (19) found in Section 2. The rest of the proof is the same as for k even, and we leave it for the reader to check the details. □

Corollary 3.2. *If* $0 \leq a \leq (n-1)/3$, *then there exists a nonsequenceable* PSTS (n) *of size* $\frac{1}{3}\binom{n}{2} - a$, *for all* $n \equiv 1 \pmod 6$ *except for* $n = 7$.

Proof. The STS (n) constructed in Theorem 3.1 has, for each point x, an almost parallel class that does not contain x. Fixing any point x_0 and removing a of the blocks in the almost parallel class that does not contain x_0 constructs a PSTS (n) of size $\frac{1}{3}\binom{n}{2} - a$ that still satisfies Theorem 1.1. □

Closing remarks

Brian Alspach gave a talk on August 9, 2018 entitled "Strongly Sequenceable Groups" at the 4th Kliakhandler Conference held at MTU. In this talk, among other things, the notion of sequencing diffuse posets was introduced and the following research problem was posed:

> "Given a triple system of order n with $\lambda = 1$, define a poset P by letting its elements be the triples and any union of disjoint triples. This poset is not diffuse in general, but it is certainly possible that P is sequenceable."

Our article shows surprisingly that there are in fact triple systems where the poset P is nonsequenceable. However, it still remains an interesting open question to construct a $\text{STS}\,(n)$ that is nonsequenceable when $n \equiv 3 \pmod 6$. A more ambitious research problem would be to determine necessary and sufficient conditions for when an $\text{STS}\,(n)$ is sequenceable.

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