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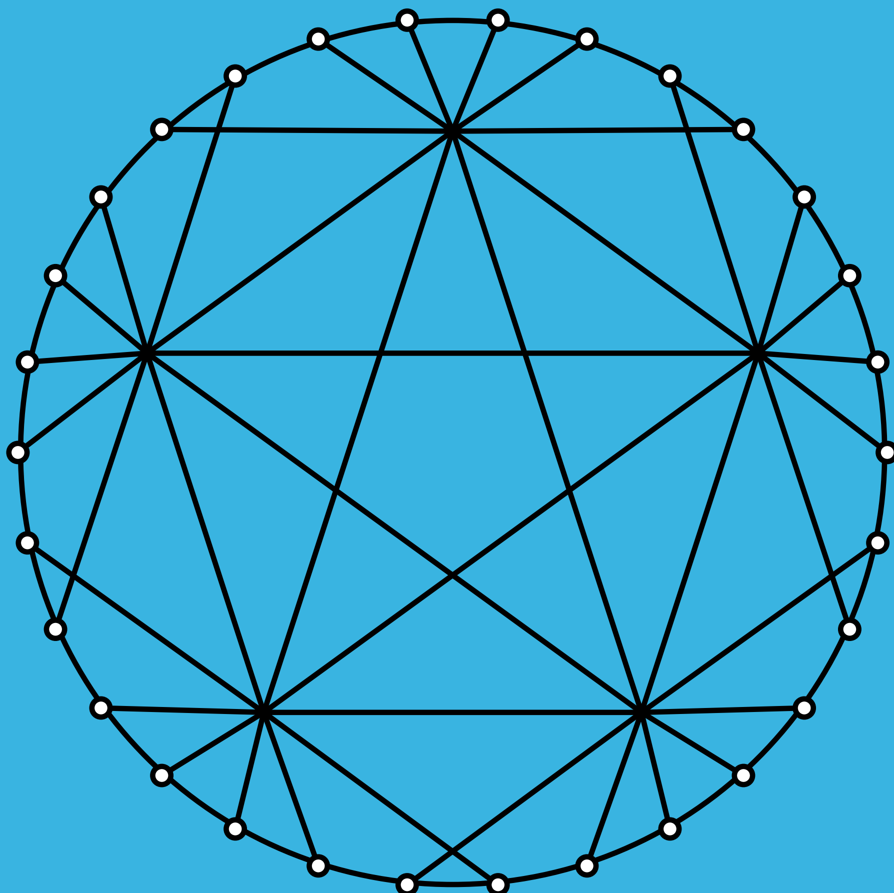

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A generalization of magic labeling of two classes of graphs

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Abstract: A *k-magic labeling* of a finite, simple graph with $|V(G)| = p$ and $|E(G)| = q$, is a bijection from the set of edges into the set of integers $\{1, 2, 3, \cdots, q\}$ such that the vertex set V can be partitioned into k sets $V_1, V_2, V_3, \cdots, V_k$, $1 \leq k \leq p$, and each vertex in the set V_i has the same vertex sum and any two vertices in different sets have different vertex sum, where a vertex sum is the sum of the labels of all edges incident with that vertex. A graph is called *k-magic* if it has a k-magic labeling. The study of k-magic labeling is very interesting, since all magic graphs are 1-magic and all antimagic graphs are p-magic. The *Splendour Spectrum* of a graph G, denoted by $SSP(G)$, is defined by $SSP(G) = \{k \mid G \text{ has a } k\text{-magic labeling}\}$.

In this paper, we determine $SSP(K_{m,n}), m$ and n are even and $SSP(T_n)$, where T_n is the friendship graph and $n \geq 1$.

1 Introduction

Let G be a finite, undirected simple connected graph with p vertices and q edges. A *magic labeling* is a bijection from the set of edges into the

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set of integers $\{1, 2, 3, \cdots, q\}$ such that all vertex sums are same, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called *magic* if it has a magic labeling. Magic labeling concept was introduced in 1963 by Sedláček [8]. An *antimagic labeling* of a graph is a bijection from the set of edges into the set of integers $\{1, 2, 3, \cdots, q\}$ such that all p vertex sum are pairwise distinct. The concept of antimagic graph was introduced by *Hartsfield and Ringel*[3] in 1990. The concept of bimagic labeling of graphs was introduced by Babujee [1] in 2004.

Motivated by the concept of magic and antimagic labeling, we define a k-magic labeling,$1 \leq k \leq p$. A *k-magic labeling* of a graph is a bijection from the set of edges into the set of integers $\{1, 2, 3, \cdots, q\}$ such that the vertex set V can be partitioned into k sets $V_1, V_2, V_3, \cdots, V_k$, $1 \leq k \leq p$, and each vertex in the set V_i has the same vertex sum and any two vertices in different sets have the different vertex sum. A graph is called *k-magic* if it has a k-magic labeling. We observe that a magic labeling is 1- magic and an antimagic labeling is p - magic so that k-magic labeling is a generalization of both magic and antimagic labeling of a graph.

We have the following problems:

1. Does there exists a graph which is k-magic for all k, $1 \leq k \leq p$?
2. Given a graph G, determine the values of k for which G is k-magic.

In this paper, we attempt to solve the above problems.

2 Construction of k-magic rectangles

A magic rectangle is an arrangement of the set of integers $\{1, 2, 3, \ldots, mn\}$ in an array of m rows and n columns so that each row adds to the same total R and each column to the same total C. The totals R and C are termed as the row magic constant and column magic constant respectively. Since the average value of set of integers $\{1, 2, 3, \ldots, mn\}$ is $A = \frac{mn+1}{2}$, we must have $R = nA$ and $C = mA$. The total of all the integers in the array is $mnA = mR = nC$. These two constants are the same just in the case $m = n$. A magic rectangle may be one of the two kinds - even by even or odd by odd. If mn is even, then $mn + 1$ is odd and so for $R = \frac{n(mn+1)}{2}$ and $C = \frac{m(mn+1)}{2}$ to be integers m and n must both be even. On the other hand, since either m or n being even would result in the product mn being even, and so if mn is odd then m and n must both be odd. In this case also R and C are integers since $mn+1$ is even. Therefore, an odd by even magic rectangle is not possible. Also, 2×2 magic rectangle is impossible.

For an update on available literature on magic rectangles we refer to Hegedorn [4] and Bier et al.[2]. In 2009, Reyes et al. [7] have provided complete solutions for constructing an even by even magic rectangle.

Motivated by the concept of magic rectangle, we define k-magic rectangle, $1 \leq k \leq m+n$.

A k-magic rectangle, $1 \leq k \leq m+n$, is a $m \times n$ arrangement of the set of integers $\{1, 2, 3, \ldots, mn\}$ so that the sums of the entries in each row and each column form a k-element set.

We observe that a magic rectangle is a 2-magic rectangle.

Now, we construct a $3, 4, 5, \ldots, (m+n)$-magic rectangle from the given 2-magic rectangle.

Let R_i and $C_j, 1 \leq i \leq m, 1 \leq j \leq n$ be the sum of all the entries in the i^{th} row and j^{th} column respectively. Let $A^{(k)}_{m\times n} = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$, be the k-magic rectangle of order $m \times n$.

Construction of $A^{(3)}_{m\times n}$ of order $m = 2s, n = 2t, s$ and t both odd, from $A^{(2)}_{m\times n}$

First we consider $A^{(2)}_{m\times n}$ be the 2-magic rectangle in [7].

In the first $\frac{n-2}{2}$ columns, $a_{1,j} - a_{1,j+1} = 2m-1$ and $a_{m-1,j+1} - a_{m-1,j} = 2m-3, 1 \leq j \leq \frac{n-2}{2}, j$ is odd. Now, we interchange the entries $a_{1,j}$ and $a_{1,j+1}$ and also interchange the entries $a_{m-1,j+1}$ and $a_{m-1,j}, 1 \leq j \leq \frac{n-2}{2}, j$ is odd.

In $(\frac{n}{2})^{th}$ and $(\frac{n+2}{2})^{th}$ columns, we have $a_{\frac{m-2}{4},\frac{n}{2}} - a_{\frac{m-2}{4},\frac{n+2}{2}} = \frac{3m+4}{2}$ and $a_{\frac{m+2}{4},\frac{n+2}{2}} - a_{\frac{m+2}{4},\frac{n}{2}} = \frac{3m}{2}$. Now, we interchange the entries $a_{\frac{m-2}{4},\frac{n}{2}}$ and $a_{\frac{m-2}{4},\frac{n+2}{2}}$ and also interchange the entries $a_{\frac{m+2}{4},\frac{n}{2}}$ and $a_{\frac{m+2}{4},\frac{n+2}{2}}$.

In the last $\frac{n-2}{2}$ columns, $a_{1,j}-a_{1,j+1}=1$ and $a_{m-1,j+1}-a_{m-1,j}=3$, $\frac{n+4}{2}\leq j\leq n, j$ is odd. Now, we interchange the entries $a_{1,j}$ and $a_{1,j+1}$ and also interchange the entries $a_{m-1,j+1}$ and $a_{m-1,j}$.

Then we get a new magic rectangle in which

$$R_j=\frac{n(mn+1)}{2}\quad \text{if } 1\leq j\leq m,$$

$$C_j=\begin{cases}\dfrac{m(mn+1)}{2}+2 & \text{if } 1\leq j\leq \dfrac{n+2}{2},\ j \text{ even}, \ \dfrac{n+4}{2}\leq j\leq n,\ j \text{ odd},\\[2mm] \dfrac{m(mn+1)}{2}-2 & \text{if } 1\leq j\leq \dfrac{n+2}{2},\ j \text{ odd}, \ \dfrac{n+4}{2}\leq j\leq n,\ j \text{ even}.\end{cases}$$

This implies that it is a 3-magic rectangle $A^{(3)}_{m\times n}$ of order $m=2s, n=2t, s$ and t both odd.

Construction of $A^{(3)}_{m\times n}$ of order $m=2s, n=2t$, where at least one of s and t is even, from $A^{(2)}_{m\times n}$

First we take the parameter s to be even without loss of generality.

We consider $A^{(2)}_{m\times n}$be the 2-magic rectangle in [7].

In $A^{(2)}_{m\times n}, a_{1,j+1}-a_{1,j}=2m-1$ and $a_{\frac{m+4}{4},j}-a_{\frac{m+4}{4},j+1}=2m-\frac{m+2}{2}, 1\leq j\leq n, j$ is odd.

Now, we interchange the entries a_{1j} and $a_{1,j+1}$ and also interchange the entries $a_{\frac{m+4}{4},j+1}$ and $a_{\frac{m+4}{4},j}$, creating a new magic rectangle in which

$$R_j=\frac{n(mn+1)}{2}\quad \text{if } 1\leq j\leq m,$$

$$C_j=\begin{cases}\dfrac{m(mn+1)}{2}+\dfrac{m}{2} & \text{if } 1\leq j\leq n,\ j \text{ is odd},\\[2mm] \dfrac{m(mn+1)}{2}-\dfrac{m}{2} & \text{if } 1\leq j\leq n,\ j \text{ is even}.\end{cases}$$

This implies that it is a 3-magic rectangle $A^{(3)}_{m\times n}$ of order $m = 2s, n = 2t$, where at least one of s and t is even, from the given $A^{(2)}_{m\times n}$.

Algorithm 2.1. *Algorithm for obtaining a* $2i$*-magic rectangle for* $2 \leq i \leq \frac{m+n}{2}$ *where* $m = 2s, n = 2t, s$ *and* t *both odd, from* $A^{(2)}_{m\times n}$.

Input: Let $A^{(2)}_{m\times n} = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$, where $m = 2s, n = 2t, s$ and t both odd be the 2-magic rectangle in [7].

Then

$$\sum_{i=1}^{m} a_{ij} = \frac{m\left(m^2+1\right)}{2}, \; 1 \leq j \leq n,$$

and

$$\sum_{i=1}^{n} a_{ij} = \frac{m\left(m^2+1\right)}{2}, \; 1 \leq j \leq m.$$

In $A^{(2)}_{m\times n}$, a_{i1}, $1 \leq i \leq m$ is of the form:

$$a_{i1} = \begin{cases} mn - i + 1 & \text{if } 1 \leq i \leq \dfrac{m}{2}, \\ m - i + 1 & \text{if } \dfrac{m+2}{2} \leq i \leq m. \end{cases}$$

Step 1: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k}{2},1}$ and $a_{m-\frac{k}{2}+1,1}$, to get a new magic rectangle in which the sum of the entries of all rows and columns are as same as in $A^{(k)}_{m\times n}$ except $R_{\frac{k}{2}}$ and $R_{m-\frac{k}{2}+1}$, where $R_{\frac{k}{2}} = \frac{n(mn+1)}{2} - (mn+1) + k$ and $R_{m-\frac{k}{2}+1} = \frac{n(mn+1)}{2} + (mn+1) - k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.This step is repeated whenever k varies from 2 to $m-2$ and k is even.

Step 2: In $A^{(m)}_{m\times n}, a_{mj}, 1 \leq j \leq n$ is of the form:

$$a_{mj} = \begin{cases} mn & \text{if } j = 1 \\ m(j-1)+1 & \text{if } 3 \leq j \leq \dfrac{n-2}{2}, \; j \text{ odd}, \; \dfrac{n}{2} \leq j \leq n, \; j \text{ even}, \\ mj & \text{if } 1 \leq j \leq \dfrac{n-2}{2}, \; j \text{ even}, \; \dfrac{n}{2} \leq j \leq n, \; j \text{ odd}. \end{cases}$$

In $A^{(m)}_{m\times n}$, interchange the entries a_{m1} and a_{mn}, to get a new magic rectangle in which $C_1 = \frac{m(mn+1)}{2} - m + 1$ and $C_n = \frac{m(mn+1)}{2} + m - 1$. It is a $(m+2)$-magic rectangle $A^{(m+2)}_{m\times n}$.

Step 3: Take k-magic rectangle and interchange the entries $a_{m,\frac{k-m+2}{2}}$ and $a_{m,n-\frac{k-m+2}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+2}{2}} = \frac{m(mn+1)}{2} + m(m+n-1-k)$ and $C_{n-\frac{k-m+2}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-1-k)$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$. This step is repeated whenever k varies from $m+2$ to $m+n-4$ and k is even.

Step 4: In $A^{(m+n-2)}_{m\times n}$, interchange the entries $a_{\frac{m}{2},n}$ and $a_{\frac{m+2}{2},n}$ and also interchange the entries $a_{m,\frac{n}{2}}$ and $a_{m,\frac{n+2}{2}}$, to get a new magic rectangle in which $C_{\frac{m}{2}} = \frac{n(mn+1)}{2} + m(n-1) - 1, C_{\frac{m+2}{2}} = \frac{n(mn+1)}{2} - m(n-1) + 1, C_{\frac{n}{2}} = \frac{m(mn+1)}{2} + 1$ and $C_{\frac{n+2}{2}} = \frac{m(mn+1)}{2} - 1$. It is a $(m+n)$-magic rectangle $A^{(m+n)}_{m\times n}$.

Output: We obtain a $4, 6, 8, \ldots, (m+n)$-magic rectangle where $m = 2s, n = 2t, s$ and t both odd, from $A^{(2)}_{m\times n}$.

Algorithm 2.2. *Algorithm for obtaining a $2i+1$-magic rectangle for $2 \le i \le \frac{m+n}{2}$ where $m = 2s, n = 2t, s$ and t both odd, from $A^{(3)}_{m\times n}$.*

Input: Let $A^{(3)}_{m\times n} = (a_{ij}), 1 \le i \le m, 1 \le j \le n$, be the 3-magic rectangle where $m = 2s, n = 2t, s$ and t both odd.

In $A^{(3)}_{m\times n}$, $a_{i1}, 1 \le i \le m$ is of the form:

$$a_{i1} = \begin{cases} m(n-2)+1 & \text{if } i = 1, \\ mn - i + 1 & \text{if } 2 \le i \le \dfrac{m}{2}, \\ m - i + 1 & \text{if } \dfrac{m+2}{2} \le i \le m,\ i \ne m-1, \\ 2m - 1 & \text{if } i = m-1. \end{cases}$$

Step 1: In $A^{(3)}_{m\times n}$, interchange the entries a_{11} and a_{m1}, to get a new magic rectangle in which $R_1 = \frac{n(mn+1)}{2} - m(n-2)$ and $R_m = \frac{n(mn+1)}{2} + m(n-2)$. It is a 5-magic rectangle $A^{(5)}_{m\times n}$.

Step 2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k+1}{2},1}$ and $a_{n-\frac{k+1}{2}+1,1}$, to get a new magic rectangle in which $R_{\frac{k+1}{2}} = \frac{n(mn+1)}{2} - mn + k$ and $R_{n-\frac{k+1}{2}+1} = \frac{n(mn+1)}{2} + mn - k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.This step is repeated whenever k varies from 5 to $m-1$ and k is odd.

Step 3: In $A^{(m+1)}_{m\times n}$, $a_{mj}, 1 \le j \le n$ is of the form:

$$a_{mj} = \begin{cases} m(n-2)+1 & \text{if } j = 1, \\ m(j-1)+1 & \text{if} \begin{cases} 3 \le j \le \dfrac{n-2}{2}, \ j \text{ is odd, or} \\ \dfrac{n}{2} \le j \le n, \ j \text{ is even;} \end{cases} \\ mj & \text{if} \begin{cases} 1 \le j \le \dfrac{n-2}{2}, \ j \text{ is even, or} \\ \dfrac{n}{2} \le j \le n, \ j \text{ is odd} \end{cases} \end{cases}.$$

In $A^{(m+1)}_{m\times n}$, interchange the entries a_{m1} and a_{mn}, to get a new magic rectangle in which $C_1 = \frac{m(mn+1)}{2} + m - 2$ and $C_n = \frac{m(mn+1)}{2} - m + 2$. It is a $(m+3)$-magic rectangle $A^{(m+3)}_{m\times n}$.

Step 4: Step 4.1 is repeated whenever k varies from $m+3$ to $m+n-3$ and $k \equiv 1(mod\ \ 4)$ and Step 4.2 is repeated whenever k varies from $m+3$ to $m+n-3$ and $k \equiv 3(mod\ \ 4)$.

Step 4.1: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+1}{2}}$ and $a_{m,n-\frac{k-m+1}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+1}{2}} = \frac{m(mn+1)}{2} + m(m+n-k) + 2$ and $C_{n-\frac{k-m+1}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-k) - 2$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Step 4.2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+1}{2}}$ and $a_{m,n-\frac{k-m+1}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+1}{2}} = \frac{m(mn+1)}{2} + m(m+n-k) - 2$ and $C_{n-\frac{k-m+1}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-k) + 2$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Output: We obtain a $5, 7, 9, \ldots, (m+n-1)$-magic rectangle where $m = 2s, n = 2t, s$ and t both odd, from the given $A^{(3)}_{m\times n}$.

Algorithm 2.3. *Algorithm for obtaining a* $4, 6, 8, \ldots, (m+n)$*-magic rectangle where* $m = 2s, n = 2t$*, where at least one of* s *and* t *is even, from* $A^{(2)}_{m\times n}$*.*

Input: Without loss of generality, we assume that s must be even.

Let $A^{(2)}_{m\times n} = (a_{ij}), 1 \le i \le m, 1 \le j \le n$, be the 2-magic rectangle where $m = 2s, n = 2t, s$ even. Then

$$\sum_{i=1}^{m} a_{ij} = \frac{m\left(m^2+1\right)}{2}, 1 \le j \le n,$$

and

$$\sum_{i=1}^{n} a_{ij} = \frac{m\left(m^2+1\right)}{2}, 1 \le j \le m.$$

In $A^{(2)}_{m\times n}$, [7] $a_{i1}, 1 \le i \le m$ is of the form:

$$a_{i1} = \begin{cases} i & \text{if } 1 \le i \frac{m}{4}, \text{ or } \frac{3m+4}{4} \le i \le, \\ mn - i + 1 & \text{if } \frac{m+4}{4} \le i \le \frac{3m}{4}. \end{cases}$$

Step 1: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k}{2},1}$ and $a_{m-\frac{k}{2}+1,1}$, to get a new magic rectangle in which $R_{\frac{k}{2}} = \frac{n(mn+1)}{2} + m + 1 - k$ and $R_{m-\frac{k}{2}+1} = \frac{n(mn+1)}{2} - m - 1 + k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$. This step is repeated whenever k varies from 2 to $\frac{m}{2}$ and k is even.

Step 2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k}{2},1}$ and $a_{m-\frac{k}{2}+1,1}$, to get a new magic rectangle in which $R_{\frac{k}{2}} = \frac{n(mn+1)}{2} - m - 1 + k$ and $R_{m-\frac{k}{2}+1} = \frac{n(mn+1)}{2} + m + 1 - k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$. This step is repeated whenever k varies from $\frac{m+4}{2}$ to $m-2$ and k is even.

Step 3: In $A^{(m)}_{m\times n}, a_{mj}, 1 \le j \le n$ is of the form:

$$a_{mj} = \begin{cases} 1 & \text{if } j = 1, \\ mj & \text{if } 2 \le j \le n,\ j \text{ is odd}, \\ m(j-1)+1 & \text{if } 1 \le j \le n,\ j \text{ is even}. \end{cases}$$

In $A^{(m)}_{m\times n}$, interchange the entries a_{m1} and a_{mn}, to get a new magic rectangle in which $C_1 = \frac{m(mn+1)}{2} + m(n-1)$ and $C_n = \frac{m(mn+1)}{2} - m(n-1)$. It is a $(m+2)$-magic rectangle $A^{(m+2)}_{m\times n}$.

Step 4: The Step 4.1 is repeated whenever k varies from $m+2$ to $m+n-4$ and $k \equiv 2(mod\ \ 4)$ and the Step 4.2 is repeated whenever k varies from $m+2$ to $m+n-4$ and $k \equiv 0(mod\ \ 4)$.

Step 4.1: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+2}{2}}$ and $a_{m,n-\frac{k-m+2}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+2}{2}} = \frac{m(mn+1)}{2} + m(m+n-k) - 1$ and $C_{n-\frac{k-m+2}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-k) + 1$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Step 4.2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+2}{2}}$ and $a_{m,n-\frac{k-m+2}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+2}{2}} = \frac{m(mn+1)}{2} + m(m+n-2-k) + 1$ and $C_{n-\frac{k-m+2}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-2-k) - 1$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Step 5: In $A^{(m+n-2)}_{m\times n}$, interchange the entries $a_{\frac{m}{2},n}$ and $a_{\frac{m+2}{2},n}$ and also interchange the entries $a_{m,\frac{n}{2}}$ and $a_{m,\frac{n+2}{2}}$, to get a new magic rectangle in which $C_{\frac{m}{2}} = \frac{n(mn+1)}{2} - 1, C_{\frac{m+2}{2}} = \frac{n(mn+1)}{2} + 1, C_{\frac{n}{2}} = \frac{m(mn+1)}{2} - 1$ and $C_{\frac{n+2}{2}} = \frac{m(mn+1)}{2} + 1$. It is a $(m+n)$-magic rectangle $A^{(m+n)}_{m\times n}$.

Output: We obtain a $4, 6, 8, \ldots, m+n$-magic rectangle where $m = 2s, n = 2t$, where at least one of s and t is even, from $A^{(2)}_{m\times n}$.

Algorithm 2.4. *Algorithm for obtaining a* $5, 7, 9, \ldots, (m+n-1)$*-magic rectangle,* $m = 2s, n = 2t$, *where at least one of* s *and* t *is even, from* $A^{(3)}_{m\times n}$.

Input: Let $A^{(3)}_{m\times n} = (a_{ij}), 1 \le i \le m, 1 \le j \le n$, be the 3-magic rectangle, $m = 2s, n = 2t$, where at least one of s and t is even.

In $A^{(3)}_{m\times n}$, $a_{i1}, 1 \le i \le m$ is of the form:

$$a_{i1} = \begin{cases} 2m & \text{if } i = 1, \\ i & \text{if } 2 \le i \le \dfrac{m}{4}, \text{ and } \dfrac{3m+4}{4} \le i \le m, \\ mn - i + 1 & \text{if } \dfrac{m+8}{2} \le i \le m, \text{ and } i \neq \dfrac{3m}{4}, \\ m(n-2) + \frac{m}{4} + 1 & \text{if } i = \dfrac{m+4}{4}. \end{cases}$$

Step 1: In $A^{(3)}_{m\times n}$, interchange the entries a_{11} and a_{m1}, to get a new magic rectangle in which $R_1 = \frac{n(mn+1)}{2} - m$ and $R_m = \frac{n(mn+1)}{2} + m$. It is a 5-magic rectangle $A^{(5)}_{m\times n}$.

Step 2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k-1}{2},1}$ and $a_{n-\frac{k-1}{2}+1,1}$, to get a new magic rectangle in which $R_{\frac{k-1}{2}} = \frac{n(mn+1)}{2} + m + 2 - k$ and $R_{n-\frac{k-1}{2}+1} = \frac{n(mn+1)}{2} - m - 2 + k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$. This step is repeated whenever k varies from 5 to $\frac{m+2}{2}$ and k is odd.

Step 3: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{\frac{k+1}{2},1}$ and $a_{n-\frac{k+1}{2}+1,1}$, to get a new magic rectangle in which $R_{\frac{k+1}{2}} = \frac{n(mn+1)}{2} - m + k$ and $R_{n-\frac{k+1}{2}+1} = \frac{n(mn+1)}{2} + m - k$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$. This step is repeated whenever k varies from $\frac{m+6}{2}$ to $m-1$ and k is odd.

Step 4: In $A^{(m+1)}_{m\times n}, a_{mj}, 1 \le j \le n$ is of the form:

$$a_{mj} = \begin{cases} 2m & \text{if } j = 1, \\ mj & \text{if } 2 \le j \le n, \text{ and } j \text{ is odd}, \\ m(j-1) + 1 & \text{if } 1 \le j \le n, \text{ and } j \text{ is even}. \end{cases}$$

In $A^{(m+1)}_{m\times n}$, interchange the entries a_{m1} and a_{mn}, to get a new magic rectangle in which $C_1 = \frac{m(mn+1)}{2} + \frac{m+2}{2} + m(n-3) + 1$ and $C_n = \frac{m(mn+1)}{2} - \frac{m}{2} - m(n-3) - 1$. It is a $(m+3)$-magic rectangle $A^{(m+3)}_{m\times n}$.

Step 5: The Step 5.1 is repeated whenever k varies from $m+3$ to $m+n-3$ and $k \equiv 3(mod\ \ 4)$ and the Step:5.2 is repeated whenever k varies from $m+3$ to $m+n-3$ and $k \equiv 1(mod\ \ 4)$.

Step 5.1: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+1}{2}}$ and $a_{m,n-\frac{k-m+1}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+1}{2}} = \frac{m(mn+1)}{2} + m(m+n+1-k) - \frac{m}{2} - 1$ and $C_{n-\frac{k-m+1}{2}+1} = \frac{m(mn+1)}{2} - m(m+n+1-k) + \frac{m}{2} + 1$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Step 5.2: Take k-magic rectangle $A^{(k)}_{m\times n}$ and interchange the entries $a_{m,\frac{k-m+1}{2}}$ and $a_{m,n-\frac{k-m+1}{2}+1}$, to get a new magic rectangle in which $C_{\frac{k-m+1}{2}} = \frac{m(mn+1)}{2} + m(m+n-1-k) + \frac{m}{2} + 1$ and $C_{n-\frac{k-m+1}{2}+1} = \frac{m(mn+1)}{2} - m(m+n-1-k) - \frac{m}{2} - 1$. It is a $(k+2)$-magic rectangle $A^{(k+2)}_{m\times n}$.

Output: We obtain a $5, 7, 9, \ldots, (m+n-1)$-magic rectangle, $m = 2s, n = 2t$, where at least one of s and t is even, from $A^{(3)}_{m\times n}$.

3 Splendour spectrum of two classes of graphs

In this section, we associate a set of positive integers to each graph G using the existence or non-existence of a k-magic labeling of G.

The *Splendour Spectrum* of a graph G, denoted by $SSP(G)$, is defined by $SSP(G) = \{k \,|\, G \text{ has a } k-\text{magic labeling}\}$. An example is provied in Figure 1.

Now, let us determine $SSP(G)$ of two classes of graphs. In [6], we proved that $SSP(K_{n,n}) = \{1, 2, \ldots, 2n\}$.

Theorem 3.1. *A complete bipartite graph $K_{m,n}, m$ and n are even, is k-magic if and only if $k \neq 1$.*

In other words, $SSP(K_{m,n}) = \{2, 3, \ldots, (m+n)\}$, where m and n are even.

Proof. Let the bipartition of $K_{m,n}$ be $r_1, r_2, \ldots, r_m$ and $c_1, c_2, \ldots, c_n$. By labeling the edge $r_i c_j$ with the contents of cell (i, j) in a $m \times n$ $k-$ magic rectangle.

In [5], Ivančo et al. proved that $K_{m,n}, m, n \geq 1, m \neq n, m, n \equiv 0(mod\ \ 2)$, is not a magic graph (1-magic graph).

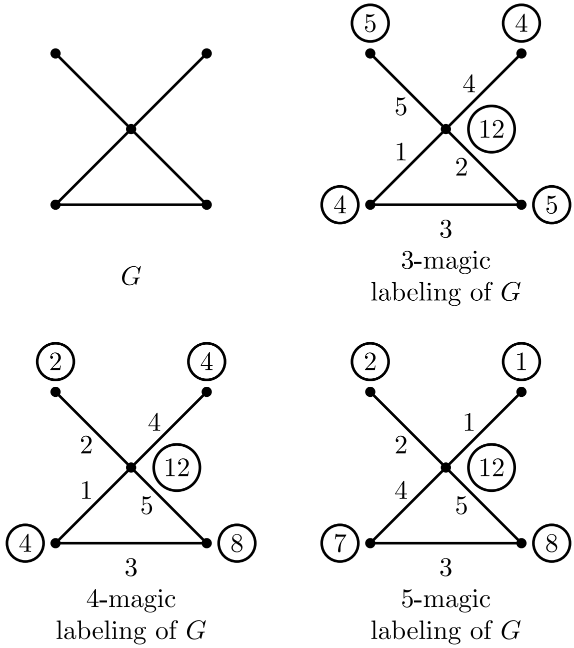

Figure 1: A graph G with $SSP(G) = \{3, 4, 5\}$.

From the constructions of 2-magic rectangle and 3-magic rectangle of order $m = 2s, n = 2t$, s and t both odd and from Algorithm 2.1 and Algorithm 2.2, we get k-magic rectangle $2 \leq k \leq m+n$ and hence $K_{m,n}$, $m = 2s, n = 2t$, s and t both odd is k-magic, $2 \leq k \leq m+n$.

From the constructions of 2-magic rectangle and 3-magic rectangle of order $m = 2s, n = 2t$, at least one of s and t even and from Algorithm 2.3 and Algorithm 2.4, we get k-magic rectangle $2 \leq k \leq m+n$ and hence $K_{m,n}$, $m = 2s, n = 2t$, at least one of s and t even is k-magic, $2 \leq k \leq m+n$.

Thus, $K_{m,n}, m$ and n are even, is k-magic if and only if $k \neq 1$. □

Open Problem 3.2. *Determine $SSP(K_{m,n})$, if either m or n is odd.*

Theorem 3.3. *A friendship graph T_n is neither magic nor 2-magic for all n.*

Proof. Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the set of vertices in $T_n, n \geq 1$ such that

v_{2n+1} is the central vertex and let $e_i = v_i v_{2n+1}, 1 \leq i \leq 2n$ and let $f_i = v_{2i-1}v_{2i}, 1 \leq i \leq n$ be the edges in $T_n, n \geq 1$.

Then we can use the edge labels from the set of integers $\{1, 2, \ldots, 3n\}$ and let S_i be the sum of the edge labels incident with the vertex $v_i, 1 \leq i \leq 2n+1$.

Now,

$$1 + 2 + \cdots + 2n \leq S_{2n+1} \leq (n+1) + (n+2) + \cdots + 3n,$$

$$1 + 2 \leq S_{2i-1} \leq (3n) + (3n-1),$$

and

$$1 + 2 \leq S_{2i} \leq (3n) + (3n-1), \ 1 \leq i \leq n.$$

This implies that

$$n(2n+1) \leq S_{2n+1} \leq n(4n+1), 3 \leq S_{2i-1} \leq 6n-1,$$

and

$$3 \leq S_{2i} \leq 6n-1, 1 \leq i \leq n.$$

Also, $6n - 1 < n(2n+1), n \geq 3$ and the vertices v_{2i-1} and v_{2i} have different vertex sum, since adjacent vertices have different magic constant, $1 \leq i \leq n$. This implies that it is at least 3-magic, $n \geq 3$.

Also, T_1, T_2 are not 1-magic and T_2 is not 2-magic, since adjacent vertices have different magic constant. By direct verification, T_2 is not 2-magic.

Thus, a friendship graph $T_n, n \geq 1$ is neither 1-magic nor 2-magic. □

Construction of 3-magic labeling of $T_n, n \geq 1$

Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the set of vertices in $T_n, n \geq 1$ such that v_{2n+1} is the central vertex and let $e_i, 1 \leq i \leq 2n$ and let $f_i, 1 \leq i \leq n$ such that $f_i = v_{2i-1}v_{2i}$ be the edges in $T_n, n \geq 1$.

Then we can use the edge labels from the set of integers $\{1, 2, \ldots, 3n\}$ and let S_i be the sum of the edge labels incident with the vertex $v_i, 1 \leq i \leq 2n+1$.

Now, we label the edges of $T_n, n \geq 1$, as follows: $e_i, 1 \leq i \leq 2n, i$ is odd as $\frac{i-1}{2} + 1$, $e_i, 1 \leq i \leq 2n, i$ is even as $n + \frac{i}{2}$ and the edge $f_i, 1 \leq i \leq n$ as $3n - i + 1$.

Then the vertex sum of the vertices

$$S_i = \begin{cases} 3n+1, & 1 \le i \le 2n, \text{ and } i \text{ odd,} \\ 4n+1, & 1 \le i \le 2n, \text{ and } i \text{ even} \end{cases}$$

$$S_{2n+1} = n(2n+1).$$

This implies that it is a 3-magic labeling of $T_n, n \ge 1$.

Construction of 4-magic labeling from 3-magic labeling of $T_n, n \ge 1$

First we consider a 3-magic labeling of T_n, $n \ge 1$.

In a 3-magic labeling, interchange the edge labels f_1 and e_1, then we get the new labeling. Then the vertex sum of the vertices

$$S_i = \begin{cases} 3n+1 & \text{if } 1 \le i \le 2n, \ i \text{ odd,} \\ n+2 & \text{if } i = 2 \\ 4n+1 & \text{if } 4 \le i \le 2n, \ i \text{ even,} \\ 2n^2+4n-1 & \text{if i = 2n+1} \end{cases}$$

This implies that it is a 4-magic labeling of T_n, $n \ge 1$.

Algorithm 3.4. *Algorithm for obtaining* $5, 7, \ldots, (2n+1)$*-magic labeling from a* 3*-magic labeling of* T_n*,* $n \ge 1$.

Input: Consider the 3-magic labeling of $T_n, n \ge 1$.

Case(i): n is odd

Step 1: Take k-magic labeling and interchange the edge labels e_{k-2} and e_{2n-k+3}, to get a new labeling in which $S_{k-2} = 5n-k+3$ and $S_{2n-k+3} = 2n+k-1$. It is a $(k+2)$-magic labeling. This step is repeated whenever k varies from 3 to n and k is odd.

Step 2: Take k-magic labeling and interchange the edge labels e_{k-n} and e_{3n-k+1}, to get a new labeling in which $S_{k-n} = 5n-k+2$ and $S_{3n-k+1} = 2n+k$. It is a $(k+2)$-magic labeling. This step is repeated whenever k varies from $n+2$ to $2n-1$ and k is odd.

Case(ii): n **is even**

- **Step 1:** Take k-magic labeling and interchange the edge labels e_{k-2} and e_{2n-k+3}, to get a new labeling in which $S_{k-2} = 5n - k + 3$ and $S_{2n-k+3} = 2n + k - 1$. It is a $(k + 2)$-magic labeling. This step is repeated whenever k varies from 3 to $n + 1$ and k is odd.
- **Step 2:** Take k-magic labeling and interchange the edge labels e_{k-n+1} and e_{3n-k+2}, to get a new labeling in which $S_{k-n-1} = 5n-k+3$ and $S_{3n-k+2} = 2n + k - 1$. It is a $(k + 2)$-magic labeling. This step is repeated whenever k varies from $n + 3$ to $2n - 1$ and k is odd.

Output: We obtain a $5, 7, \ldots, (2n+1)$-magic labeling from 3-magic labeling of T_n, $n \geq 1$.

Algorithm 3.5. *Algorithm for obtaining* $6, 8, \ldots, 2n$*-magic labeling from 4-magic labeling of* $T_n, n \geq 1$.

Input: Consider the 4-magic labeling of T_n, $n \geq 1$.

Step 1: In a 4-magic labeling interchange the edge labels e_1 and e_{2n}, to get a new labeling in which $S_1 = 2n + 1$ and $S_{2n} = 5n + 1$. It is a 6-magic labeling.

Case(i): n **is odd.**

- **Step 2:** Take a k-magic labeling and interchange the edge labels e_{k-3} and e_{2n-k+4}, to get a new labeling in which $S_{k-3} = 5n - k + 4$ and $S_{2n-k+4} = 2n+k-2$. This implies that it is a $(k+2)$-magic labeling. This step is repeated whenever k varies from 6 to $n+1$ and k is even.
- **Step 3:** Take a k-magic labeling and interchange the edge labels e_{k-n+1} and e_{3n-k+2}, to get a new labeling in which $S_{k-n+1} = 5n - k + 2$ and $S_{3n-k+2} = 2n + k$. This implies that it is a $(k + 2)$-magic labeling. This step is repeated whenever k varies from $n + 3$ to $2n - 2$ and k is even.

Case(ii): n **is even.**

- **Step 2:** Take k-magic labeling and interchange the edge labels e_{k-3} and e_{2n-k+4}, to get a new labeling in which $S_{k-3} = 5n - k + 4$

and $S_{2n-k+4} = 2n+k-2$. This implies that it is a $(k+2)$-magic labeling. This step is repeated whenever k varies from 6 to $n+2$ and k is even.

Step 3: Take k-magic labeling and interchange the edge labels e_{k-n} and e_{3n-k+3}, to get a new labeling in which $S_{k-n} = 5n-k+2$ and $S_{3n-k+3} = 2n+k$. It is a $(k+2)$-magic labeling. This step is repeated whenever k varies from $n+4$ to $2n-2$ and k is even.

Output: We obtain a $6, 8, \ldots, 2n$-magic labeling from 4-magic labeling of T_n, $n \geq 1$.

Theorem 3.6. *The friendship graph T_n, $n \geq 1$ is k-magic if and only if $k \neq 1, 2$.*

In other words $SSP(T_n) = \{3, 4, \ldots, 2n+1\}$.

Proof. From Theorem 3.3, T_n is neither magic nor 2-magic.

From the constructions of 3-magic labeling and 4-magic labeling and from Algorithm 3.4 and Algorithm 3.5, we get k-magic labeling of $T_n, n \geq 1, 2 \leq k \leq 2n+1$ and hence T_n, $n \geq 1$, is k-magic if and only if $k \neq 1, 2$.

□

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