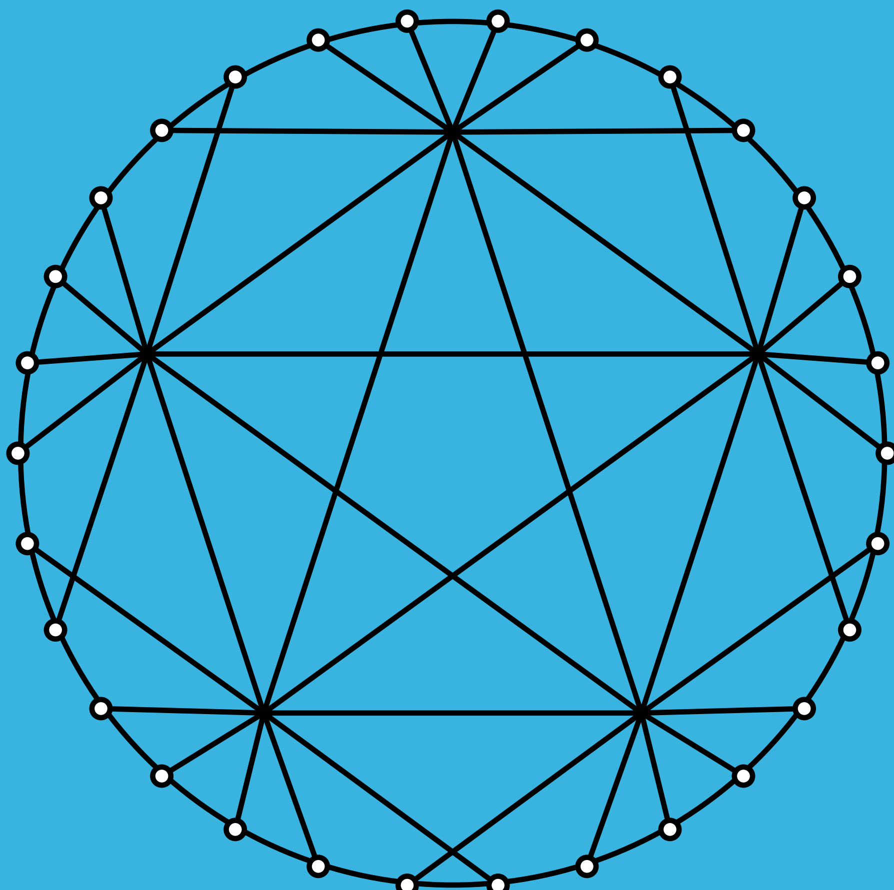
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 82
February 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **82** (2018), Pages 51–55

Not every bipartite double cover is canonical

Tomaž Pisanski

University of Primorska, FAMNIT, Koper, Slovenia and IMFM, Ljubljana, Slovenia.
Tomaz.Pisanski@upr.si.

Abstract: It is explained why the term *bipartite double cover* should not be used to designate canonical double cover alias Kronecker cover of graphs.

Keywords: Kronecker cover, canonical double cover, bipartite double cover, covering graphs.

Math. Subj. Class. (2010): 05C76, 05C10.

It is not uncommon to use different terminology or notation for the same mathematical concept. There are several reasons for such a phenomenon.

• Authors independently discover the same object and have to name it. It is quite natural that they choose different names.

• Sometimes the same concept appears in different mathematical schools or in different disciplines. By using particular terminology in a given context makes understanding of such a concept much easier.

• Sometimes original terminology is not well-chosen, not intuitive and it is difficult to relate the name of the object to its meaning. A name that is more appropriate for a given concept is preferred.

One would expect terminology to be as simple as possible, and as easy to connect it to the concept as possible. However, it should not be too simple.

Received: 8 October 2018
Accepted: 21 January 2018

In other words it should not introduce ambiguity. Unfortunately, the term *bipartite double cover* that has been used lately in several places to replace an older term *canonical double cover*, also known as the *Kronecker cover*, [1, 4], is ambiguous.

Let X be a graph. Its Kronecker cover KC(X) is the tensor product [3] of X by K_2. This means that the adjacency matrix of KC(X) is the tensor product of the adjacency matrices of X and K_2. Since the tensor product of matrices is also known as the Kronecker product, the term Kronecker cover seems to be appropriate.

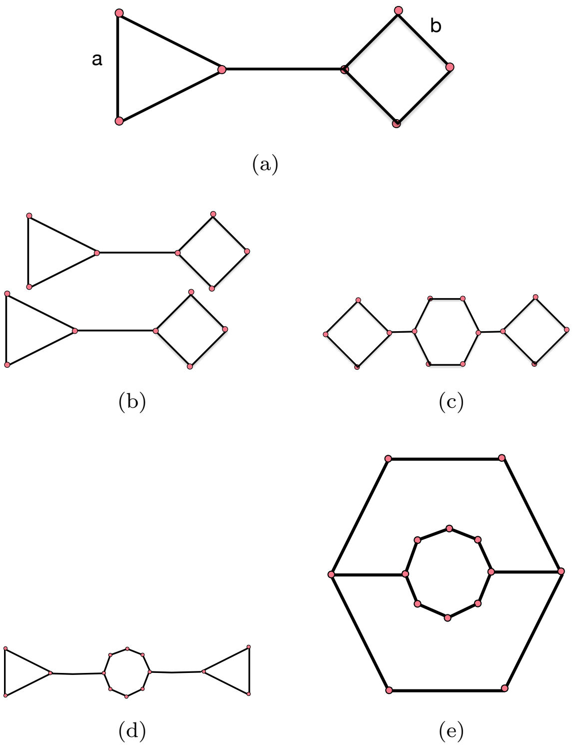


Figure 1: Graph B with at most two non-zero voltages a, b. Unlabeled edges carry voltage 0 (a). Double cover of B for $a = 0, b = 0$ (b). Double cover of B for $a = 1, b = 0$, Kronecker cover (c). Double cover of B for $a = 0, b = 1$ (d). Double cover of B for $a = 1, b = 1$ (e).

The term *canonical double cover*, which is a synonym for the Kronecker cover, has a different intuitive motivation which does not come from graph products but from covering graph theory. The topic of this note considers covering graphs which are combinatorial analogs of covering spaces from algebraic topology. For basic notions of covering graphs and voltage graphs the reader is referred to the classical book [2] and to [5]. A mapping $p : X \to B$ is a graph covering projection, B being the *base graph* and X being the *covering graph,* if it is a local isomorphism. It turns out that for connected base graph B there exist a natural number m such that each preimage of p, called the *fibre* has the same number of elements; in this case X is called an *m-fold cover* over B. Two-fold covers are also known as *double covers.* Each double cover of B can be described by 0-1 labelling of edges of B. These labels are called *voltages.* The graph X is then constructed in two layers. Each vertex v of B gives rise to two vertices v_0 and v_1 of X. If voltage 0 is assigned to an edge uv of B the edge gives rise to two edges u_0v_0 and u_1v_1. If voltage 1 is assigned to uv the corresponding fibre consists of edges u_0v_1 and u_1v_0. It turns out that double covers depend only on the net voltages (parities) around cycles of B. The *trivial double cover* has all voltages equal to 0. The corresponding double cover consists of two copies of B, ie., it is $2B$. On the opposite side the double cover obtained by all 1 voltages is called the *canonical double cover.* One of the main features of voltage graphs is the fact that two voltage assignments give rise to the same covering graph with the same covering projection if and only if each cycle of the base graph has the same net voltage relative to both voltage assignments. Such voltage assignments and corresponding covers are called *equivalent.* It is not hard to see that canonical double cover is bipartite and isomorphic to the Kronecker cover. Both the trivial double cover and the canonical double cover can be defined in terms of graph B alone. In a similar way the clone cover defined and studied in [6] is uniquely determined by B itself.

Consider the graph B with voltages a and b as in Figure 1(a). There are four non-isomorphic double covers. The trivial cover (b) is disconnected. Both (d) and (e) are non-bipartite double covers only (c) is bipartite and also Kronecker cover.

One can show the following:

Proposition 1. *Let B be connected non-bipartite graph. Then among double covers only the Kronecker cover $KC(B)$ is bipartite.*

Proof. It follows from the definition that the Kronecker cover is bipartite. Now take an arbitrary bipartite double cover X of a non-bipartite graph B. Since the cover is bipartite every odd cycle C of the base graph B must have net voltage 1 and unwind to a cycle of double length. Suppose that there is an even cycle D with net voltage 1 so that D unwind, too. Let P be a path joining C to D. By starting at the vertex common to C and P, traversing first C then P, following D and then P in the opposite direction, the resulting closed walk W has odd length but net voltage 0. It follows that the covering graph is non-bipartite: a contradiction. Hence all even cycles (if they exist) must have net voltage 0. But then by switching all 0 voltages to 1, does not change the covering graph X, which, in turn is the canonical cover: $X = KC(B)$. $\square$

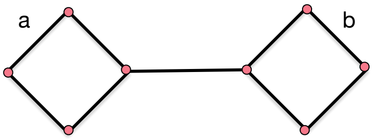


Figure 2: Bipartite graph X with at most two non-zero voltages a, b. Unlabeled edges carry voltage 0. It gives rise to three nonisomorphic bipartite double covers: $a = 0, b = 0$ gives rise to the trivial cover that is simultaneously the Kronecker cover, $a = 1, b = 0$ and $a = 0, b = 1$ result in isomorphic covers, different from the cover for $a = 1, b = 1$.

Proposition 2. *There exist disconnected non-bipartite graphs with bipartite double covers different from their Kronecker covers.*

Proof. Let $B = C_3 \cup C_4$. Then $KC(B) = C_6 \cup 2C_4$. By assigning net voltage 1 to each cycle, the resulting double cover $C_6 \cup C_8$ is clearly bipartite, too. $\square$

Proposition 3. *Every double cover of bipartite graph is bipartite. Among double covers of a connected bipartite graph only the Kronecker cover is disconnected.*

Proof. Every double cover is regular. Hence the only disconnected double cover over connected graph is the trivial cover. The Kronecker cover over a bipartite graph is trivial. $\square$

An example associated with this Proposition is depicted in Figure 2.

As we have seen the term bipartite double cover correctly designates the canonical double cover only in case the base graph is connected and non-bipartite. If either of the two conditions is dropped, i.e. if the base graph is disconnected or bipartite, the term *bipartite double cover* becomes ambiguous.

Acknowledgements

The author would like to thank Aleksander Malnič for careful reading of this note. Supported in part by the ARRS Grant P1-0294. Remarks made by anonymous referees further improved the presentation and are gratefully acknowledged.

References

[1] G. Gévay, T. Pisanski, Kronecker covers, V-construction, unit-distance graphs and isometric point-circle configurations, *Ars Math. Contemp.* **7** (2014), 317–336.

[2] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

[3] W. Imrich and S. Klavžar, *Product graphs*, Wiley-Interscience, New York, 2000.

[4] W. Imrich and T. Pisanski, Multiple Kronecker covering graphs, *European J. Combin.* **29** (2008), 1116–1122.

[5] A. Malnič, R. Nedela, M. Škoviera, Lifting graph automorphisms by voltage assignments, *European J. Combin.* **21** (2000), 927–947.

[6] A. Malnič, T. Pisanski, A. Žitnik, The clone cover, *Ars Math. Contemp.* **8** (2015), 95–113.