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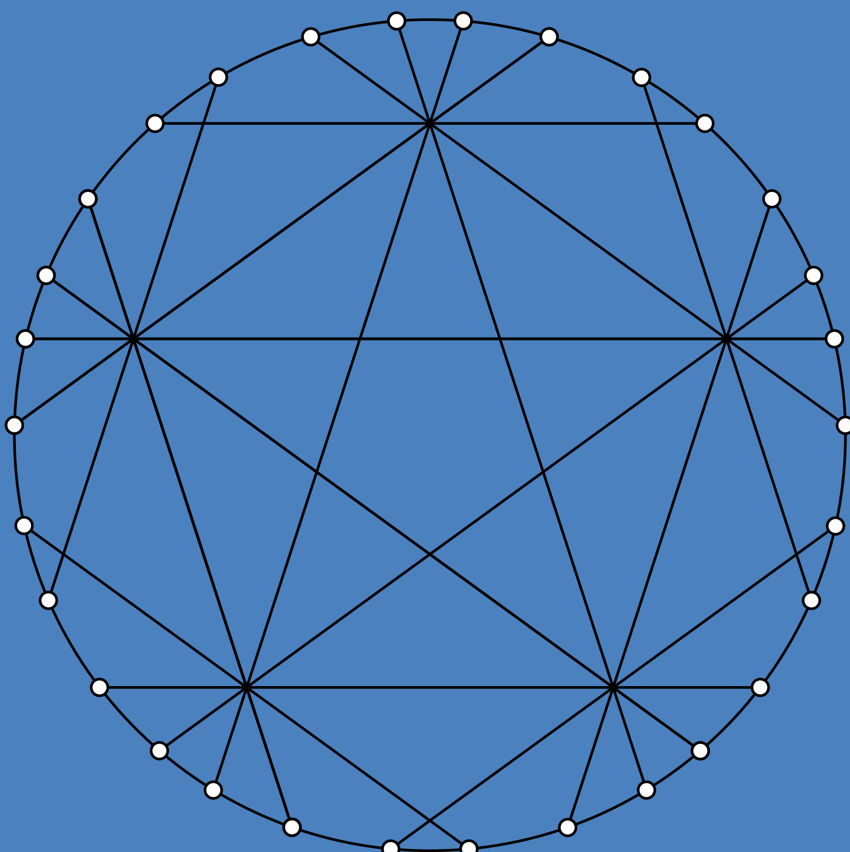

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Theta Graphs are Hall t-chromatic for all $t = 0, 1, 2, \ldots$

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Abstract: It is shown that every graph made up of internally disjoint paths joining two vertices is Hall t-chromatic for all non-negative integers t.

1 Introduction

Let G be a simple, finite graph with the vertex set, edge set, vertex independence number, chromatic number and fractional chromatic number of G denoted $V(G)$, $E(G)$, $\alpha(G)$, $\chi(G)$, $\chi_f(G)$, respectively; for definitions of these terms see [12]. The *Hall ratio* of G is

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} :\ H \text{ is a subgraph of } G\right\}.$$

Let $\mathbb{N}$ be the set of non-negative integers.

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For any graph G, a function $\kappa : V(G) \to \mathbb{N}$ is called a *color demand* on, or for, G. Let C be an infinite set of colors. A function $L : V(G) \to \{\text{finite subsets of } C\} = \mathcal{F}(C)$ is called a *color supply* for, or a *list assignment* to, G. For a color supply L and a color demand κ for G, a *proper (L, κ) coloring* of G is a function $\varphi : V(G) \to \mathcal{F}(C)$ satisfying, for all $u, v \in V(G)$:

(i) $|\varphi(v)| = \kappa(v)$;

(ii) If $uv \in E(G)$, then $\varphi(u) \cap \varphi(v) = \emptyset$;
[Equivalently; for each $\sigma \in C$, $\{v \in V(G) \mid \sigma \in \varphi(v)\}$ is an independent set of vertices in G.]

(iii) $\varphi(v) \subseteq L(v)$.

Suppose that L is a color supply, and κ is a color demand, for G, and H is a subgraph of G. For $\sigma \in C$, let $H(\sigma, L)$ be the subgraph of H induced by $\{v \in V(H) \mid \sigma \in L(v)\}$. [The null graph, with no edges nor vertices, is allowed to exist in this paper.] G, L, κ satisfy *Hall's condition* if and only if for each subgraph H of G,

$$\sum_{\sigma \in C} \alpha\big(H(\sigma, L)\big) \ \geq \sum_{v \in V(H)} \kappa(v) \qquad (*)_H$$

Clearly G, L, and κ satisfy Hall's condition if $(*)_H$ holds for every connected induced subgraph H of G.

Hall's condition on G, L and κ is a necessary condition for the existence of a proper (L, κ) coloring of G. The name of this condition descends from the fact that when G is a complete graph, Hall's condition on G, L, and κ is sufficient for the existence of a proper (L, κ) coloring of G; this assertion is a restatement of the extension of Hall's Theorem [6] to the question of the existence of pairwise disjoint subset representatives of prescribed cardinalities of given sets. The extension is due to Rado [10], Halmos and Vaughan [7], and possibly others. For a fuller discussion of these matters see [2] or [8].

The question of for which G there is a proper (L, κ) coloring of G whenever G, L and κ satisfy Hall's condition is answered completely in [3]. We will need only the following special case, which is also proven in [2].

Theorem 1.1 (Path Theorem)**.** *If P is a finite path, L is a color supply for P, κ is a color demand on P, and P, L, and κ satisfy Hall's condition, then there is a proper (L, κ) coloring of P.*

We consider a special case of Hall's condition where $L(v)$ is the same t-element subset for all $v \in V(G)$: $L = \{1, ..., t\} = [t]$. With this constant assignment L, Hall's condition becomes the following, which we will call *Hall's t-condition* on κ, G: for each subgraph H of G

$$\sum_{\sigma=1}^{t} \alpha\big(H(\sigma, L)\big) = t\alpha(H) \geq \sum_{v \in V(H)} \kappa(v) \qquad (**)_H$$

As for Hall's condition in general, for G and κ to satisfy Hall's t-condition it suffices that $(**)_H$ holds for all connected induced subgraphs H of G.

For $t \in \mathbb{N}$ (when $t = 0, [t] = L = \emptyset$) and a color demand κ for G, a *proper* (t, κ) *coloring* of G is a function $\varphi : V(G) \to 2^{[t]}$ satisfying, for all $u, v \in V(G)$:

(i) $|\varphi(v)| = \kappa(v)$;

(ii) if $uv \in E(G)$, then $\varphi(u) \cap \varphi(v) = \emptyset$.

A graph G is said to be *Hall t-chromatic* if the only color demands κ for which there does not exist a proper (t, κ) coloring of G are those that fail Hall's t-condition. In other words, G is Hall t-chromatic if and only if for all κ, Hall's t-condition on κ and G is sufficient for the existence of a proper (t, κ) coloring of G. The *Hall t-chromatic spectrum* of G is $\tau(G) = \{t \in \mathbb{N} \mid G \text{ is Hall } t\text{-chromatic}\}$. It is known from previous results in [1] and [4] that, for all finite simple graphs G,

(i) $\{0, 1, 2\} \subseteq \tau(G)$;

(ii) if H is an induced subgraph of G, then $\tau(G) \subseteq \tau(H)$;

(iii) if $\tau(G)$ is an infinite set, then

$$\chi_f(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} : H \text{ is a subgraph of } G\right\} = \rho(G).$$

We think the result (iii) is reason enough to further pursue the study of the Hall t-chromatic spectra of graphs. There are also intriguing questions about Hall t-chromatic spectra about which we know very little. For instance: Is $\tau(G)$ always a block of consecutive integers, either $\mathbb{N}$ or $\{0, ..., N\}$

for some $N \geq 2$? The only known values of $\tau(G)$ are $\mathbb{N}$ and $\{0, 1, 2\}$.

Another question: Is the converse of (iii), above, true? That is, does $|\tau(G)| < \infty$ imply that $\chi_f(G) > \rho(G)$? (It is well known that $\chi_f(G) \geq \rho(G)$ for all G; see [11].)

Our purpose here is nowhere near so lofty as these mysteries. We aim simply to show that $\tau(G) = \mathbb{N}$ for all graphs in a certain class, the class of theta graphs. A *theta graph* is a union of $m \geq 3$ paths, internally disjoint with the same end vertices. The term theta graph arises from the fact that such a union of $m = 3$ paths looks like the Greek letter theta. Let the set of all theta graphs which are the union of m internally disjoint paths with common end vertices be denoted Θ_m.

We will need the following results from [1] and [4].

Lemma 1.2.

(a) If G is a cycle, then $\tau(G) = \mathbb{N}$.

(b) If G is bipartite, then $\tau(G) = \mathbb{N}$.

(c) If $t \in \tau(G_i)$ for $i = 1, 2$, and $G_1 \cap G_2$ is a clique, then $t \in \tau(G_1 \cup G_2)$.

The following is an easy corollary of Lemma 1.2.

Corollary 1.3.

(a) If G is a theta graph and one of the internally disjoint paths whose union is G is a single edge, then $\tau(G) = \mathbb{N}$.

(b) If G is a theta graph and the internally disjoint paths whose union is G are either all of even length or all of odd length, then $\tau(G) = \mathbb{N}$.

Proof. For part (a), let $G \in \Theta_m$, $m \geq 3$ and suppose that one of the paths comprising G is a single edge. We proceed by induction on m. If $m = 3$, then $G = G_1 \cup G_2$ where G_1, G_2 are cycles and $G_1 \cap G_2 = K_2$. By (a) and (c) of Lemma 1.2, it follows $\tau(G) = \mathbb{N}$.

Now suppose $m > 3$. Then $G = G_1 \cup G_2$ where $G_1 \in \Theta_{m-1}$ and one of its constituent paths is a single edge, G_2 is a cycle, and $G_1 \cap G_2 = K_2$. Then $\tau(G) = \mathbb{N}$ follows from the induction hypothesis and (a) and (c) of Lemma 1.2.

If the hypothesis of (b) holds, then G is bipartite, so the conclusion of (b) holds by Lemma 1.2. □

2 Main Result and proof

Lemma 2.1. *If $t, a, b, c \in \mathbb{N}$, $a \geq b \geq c$ and $t \geq a + b - c$, then there exist $A, B \subseteq [t]$ such that $|A| = a$, $|B| = b$, and $|A \cap B| = c$.*

Proof. Let $a \geq b \geq c$, and $t \geq a + b - c \geq a$. Take an a-subset, A, of $[t]$. Let $Z \subseteq [t] \setminus A$, such that $|Z| = b - c$. Let $W \subseteq A$ such that $|W| = c$. Let $B = W \cup Z$. Then $|B| = c + b - c = b$ and $|A \cap B| = |W| = c$. □

Theorem 2.2. *Let G be a graph that consists of $m \geq 3$ internally disjoint paths joining a vertex $u \in V(G)$ and a vertex $v \in V(G)$. Then G is Hall t-chromatic for all $t \in \mathbb{N}$.*

Proof. By Corollary 1.3, we may assume all of the paths joining u and v are of lengths > 1, at least one of those lengths is odd, and at least one is even.

Given an integer $t > 2$ and $\kappa : V(G) \to \mathbb{N}$ such that G and κ satisfy Hall's t-condition: for each choice of paths $u, x_1, x_2, ..., x_{2p}, v$ and $u, y_1, y_2, ..., y_{2q-1}, v$, respectively of odd and even lengths ≥ 2, we have, by $(**)_H$, where H is the odd cycle which is the union of these two paths:

$$t(p + q) \geq \kappa(u) + \kappa(v) + \sum_{i=1}^{2p} \kappa(x_i) + \sum_{j=1}^{2q-1} \kappa(y_j). \tag{2.1}$$

Let

$$s = \min \left\{ tp - \sum_{i=1}^{2p} \kappa(x_i) : \; u, P_x, v = u, x_1, ..., x_{2p}, v \text{ is a } u\text{-}v \text{ path of } G \right\}.$$

For any such path $P_x = x_1, ..., x_{2p}$, we have $t\alpha(P_x) = tp \geq \sum_{i=1}^{2p} \kappa(x_i)$, so s is a non-negative integer.

Without loss of generality, we can assume $\kappa(u) \geq \kappa(v)$. There are two cases to consider. In Case I, $s \in \{0, \ldots, \kappa(v)\}$. In Case II, $s > \kappa(v)$.

Case I. $s \in \{0, ..., \kappa(v)\}$. We have that $s \leq \kappa(v) \leq \kappa(u)$. By Lemma 2.1, we can find $\varphi(u), \varphi(v) \subseteq [t]$ such that $|\varphi(u)| = \kappa(u)$, $|\varphi(v)| = \kappa(v)$, and $|\varphi(u) \cap \varphi(v)| = s$, provided

$$t \geq \kappa(u) + \kappa(v) - s. \tag{2.2}$$

For some path $P = x_1, x_2, ..., x_{2p}$, we have $tp - \sum_{i=1}^{2p} \kappa(x_i) = s$.
Let uPv be the subgraph of G induced by u, v, and the vertices of P.
From $(**)_{uPv}$, we get

$$\begin{aligned} t(p+1) &\geq \kappa(u) + \kappa(v) + \sum_{i=1}^{2p} \kappa(x_i) \Rightarrow \\ t &\geq \kappa(u) + \kappa(v) - (tp - \sum_{i=1}^{2p} \kappa(x_i)) \\ &= \kappa(u) + \kappa(v) - s, \end{aligned}$$

so (2.2) holds.

Therefore, by Lemma 2.1, we have $\varphi(u), \varphi(v) \subseteq [t]$ satisfying $|\varphi(u)| = \kappa(u)$, $|\varphi(v)| = \kappa(v)$, and $|\varphi(u) \cap \varphi(v)| = s$.

Define a list assignment, L, to $G - \{u, v\}$ by:

$L(w) = [t] \setminus \varphi(u)$ if w is adjacent to u and not to v;

$L(w) = [t] \setminus \varphi(v)$ if w is adjacent to v and not to u;

$L(w) = [t] \setminus (\varphi(u) \cup \varphi(v))$ if w is adjacent to u and to v;

$L(w) = [t]$ otherwise.

We want to show that $G - \{u, v\}$, L, κ satisfy Hall's condition; this will show that each component of $G - \{u, v\}$ has a proper (L, κ) coloring, due to the Path Theorem. This will be sufficient to show G has a proper (t, κ) coloring due to the way L is defined.

To show that $P_x = x_1, x_2, ..., x_{2p}$ satisfies Hall's condition with L and κ, the only subgraph we need to check is P_x itself, as all proper subgraphs of P_x are easily shown to satisfy this condition. To see this, observe that the path $u, x_1, ..., x_{2p-1}$ has a proper (t, κ) coloring, by the Path Theorem, because G and κ satisfy Hall's t-condition, and this condition applies to every subgraph of G. We may assume that u is colored with $\varphi(u)$ in this coloring, since the choice of the set A in Lemma 2.1 is arbitrary. Therefore, this coloring restricted to $x_1, ..., x_{2p-1}$ is a proper (L, κ) coloring of that path, and therefore, that path satisfies Hall's condition with L and κ. Similarly, $x_2, ..., x_{2p}$ satisfies Hall's condition with L and κ. Every subpath of P_x which is not the whole path is a subpath of either $P_x - x_{2p}$ or of $P_x - x_1$. Therefore, we need only show that $(*)_{P_x}$ is satisfied, to show that P_x, L, and κ satisfy Hall's condition. A similar argument applies to paths $y_1, ..., y_{2q-1}$.

We need to show that $\sum_{\sigma=1}^{t} \alpha\big(P_x(\sigma, L)\big) \geq \sum_{i=1}^{2p} \kappa(x_i)$.

If $\sigma \in [t] \setminus \big(\varphi(u) \cup \varphi(v)\big)$, then $\alpha\big(P_x(\sigma, L)\big) = p$.

If $\sigma \in \big(\varphi(u) \setminus \varphi(v)\big) \cup \big(\varphi(v) \setminus \varphi(u)\big)$, then $\alpha\big(P_x(\sigma, L)\big) = p$.

If $\sigma \in \varphi(u) \cap \varphi(v)$, then $\alpha\big(P_x(\sigma, L)\big) = p - 1$.

Therefore,

$$\begin{aligned}
\sum_{\sigma=1}^{t} \alpha\big(P_x(\sigma, L)\big) &= p\big(t - |\varphi(u) \cup \varphi(v)|\big) \\
&\quad + p\big(|\varphi(u) \cup \varphi(v)| - |\varphi(u) \cap \varphi(v)|\big) \\
&\quad + \big(p - 1\big)|\varphi(u) \cap \varphi(v)| \\
&= p(t - s) + s(p - 1) \\
&= pt - s \geq \sum_{i=1}^{2p} \kappa(x_i)
\end{aligned}$$

by the definition of s.

Given $P_y = y_1, ..., y_{2q-1}$, a component of $G - \{u, v\}$ of odd order: we aim to show that $\sum_{\sigma=1}^{t} \alpha\big(P_y(\sigma, L)\big) \geq \sum_{j=1}^{2q-1} \kappa(y_j)$. There are 2 subcases to consider. In Subcase Ia, $q > 1$. In Subcase Ib, $q = 1$.

Subcase Ia. $q > 1$.

If $\sigma \in [t] \setminus \big(\varphi(u) \cup \varphi(v)\big)$, then $\alpha\big(P_y(\sigma, L)\big) = q$.

If $\sigma \in \big(\varphi(u) \setminus \varphi(v)\big) \cup \big(\varphi(v) \setminus \varphi(u)\big)$, then $\alpha\big(P_y(\sigma, L)\big) = q - 1$.

If $\sigma \in \varphi(u) \cap \varphi(v)$, then $\alpha\big(P_y(\sigma, L)\big) = q - 1$.

Therefore,

$$\begin{aligned}\sum_{\sigma=1}^{t} \alpha\big(P_y(\sigma, L)\big) &= q\big(t - |\varphi(u) \cup \varphi(v)|\big) + \big(q - 1\big)|\varphi(u) \cup \varphi(v)| \\ &= qt - |\varphi(u) \cup \varphi(v)| \\ &= qt - (\kappa(u) + \kappa(v) - s) \\ &= qt - \kappa(u) - \kappa(v) + s.\end{aligned}$$

We want:

$$qt - \kappa(u) - \kappa(v) + s \geq \sum_{j=1}^{2q-1} \kappa(y_j). \tag{2.3}$$

By the definition of s, we can find in $G - \{u, v\}$ a component, $P_x = x_1, ..., x_{2p}$, such that $tp = s + \sum_{i=1}^{2p} \kappa(x_i)$. If we plug this into inequality (2.1), then we get:

$$\begin{aligned}t(p + q) &= tq + s + \sum_{i=1}^{2p} \kappa(x_i) \\ &\geq \kappa(u) + \kappa(v) + \sum_{i=1}^{2p} \kappa(x_i) + \sum_{j=1}^{2q-1} \kappa(y_j),\end{aligned}$$

which implies (2.3).

Subcase Ib. $q = 1$. In this subcase, $P_y = y_1$. The list assignment in this case is $L(y_1) = [t] \setminus \big(\varphi(u) \cup \varphi(v)\big)$. Then

$$\begin{aligned}\sum_{\sigma=1}^{t} \alpha\big(P_y(\sigma, L)\big) &= |L(y_1)| \\ &= t - |\varphi(u) \cup \varphi(v)| \\ &= t - (\kappa(u) + \kappa(v) - s).\end{aligned}$$

By a similar argument to the first subcase, we see that P_y satisfies the inequality, $t - \kappa(u) - \kappa(v) + s \geq \kappa(y_1)$.

Case II. $s > \kappa(v)$. For this case we take $\varphi(u) = \{1, ..., \kappa(u)\}$ and $\varphi(v) = \{1, ..., \kappa(v)\}$. Let L be the list assignment to $G - \{u, v\}$ as defined in Case I. As in Case I, we are done if we show that the inequality $(*)_H$ holds for each maximal path H in $G - \{u, v\}$. The inequality for Hall's condition is satisfied by L and κ on each of the internal paths P_x of even order, $x_1, ..., x_{2p}$, by the following argument.
If $\sigma \in [t] \setminus \big(\varphi(u) \cap \varphi(v)\big) = \{\kappa(v) + 1, ..., t\}$, then $\alpha\big(P_x(\sigma, L)\big) = p$.
If $\sigma \in \varphi(u) \cap \varphi(v) = \{1, ..., \kappa(v)\}$, then $\alpha\big(P_x(\sigma, L)\big) = p - 1$. Therefore,

$$\begin{aligned}\sum_{\sigma=1}^{t} \alpha\big(P_x(\sigma, L)\big) &= p\big(t - \kappa(v)\big) + (p-1)\kappa(v) \\ &= pt - \kappa(v) \\ &> pt - s \\ &\geq pt - \left[tp - \sum_{i=1}^{2p} \kappa(x_i)\right] \\ &= \sum_{i=1}^{2p} \kappa(x_i).\end{aligned}$$

But the same method does not show that the inequality for Hall's condition is satisfied by L and κ for path components of $G - \{u, v\}$ of odd order, $y_1, ..., y_{2q-1}$.

For $P_y = y_1, ..., y_{2q-1}$, we will give a proper (L, κ) coloring of the path. Color as follows:

$$\varphi(y_j) = \begin{cases} \{t - \kappa(y_j) + 1, ..., t\} & \text{if } j \text{ is odd} \\ \{1, ..., \kappa(y_j)\} & \text{if } j \text{ is even.} \end{cases}$$

This is a proper (L, κ) coloring of $y_1, ..., y_{2q-1}$ because the path $u, y_1, ..., y_{2q-1}, v$ satisfies Hall's t-condition with κ, and therefore $t = t\alpha(zw) \geq \kappa(z) + \kappa(w)$ for each edge zw of that path.

Hence G has a proper (t, κ) coloring. Since κ was arbitrary, it follows that G is Hall t-chromatic. □

Corollary 2.3. *If G is a theta graph, then $\chi_f(G) = \rho(G)$.*

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