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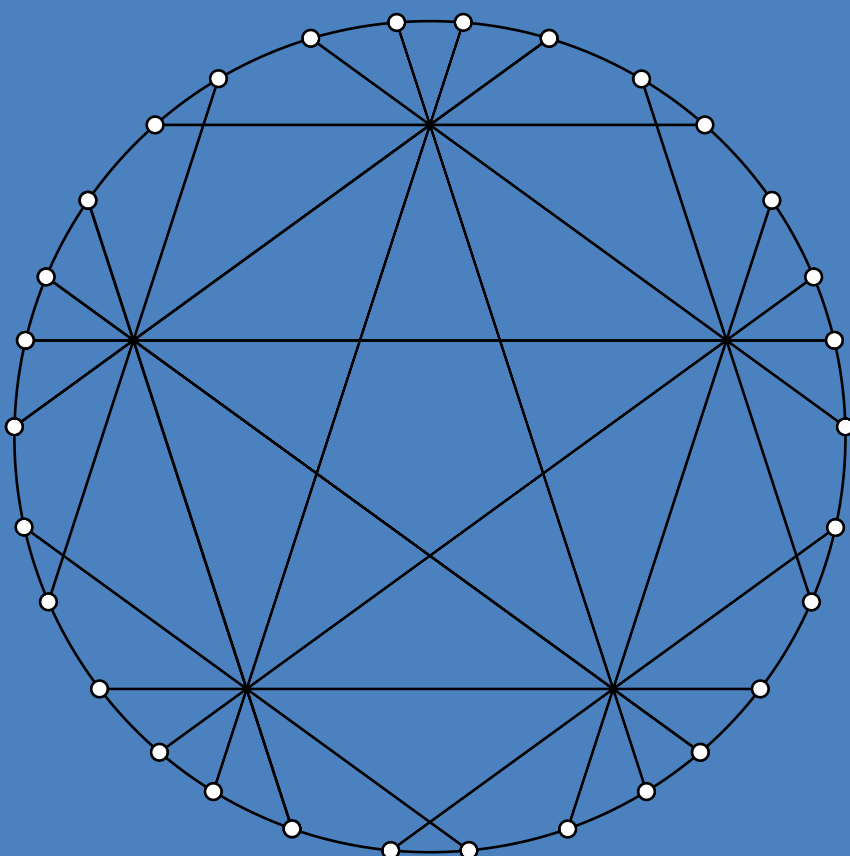

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The Graphs that Dahlhaus Called "Good Generalized Strongly Chordal"

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Abstract

In a 1993 technical report, Elias Dahlhaus described what he called "good generalized strongly chordal graphs," a natural strengthening of a generalization of the standard strengthening of chordal graphs. We present a new, more conventional characterization of this proposed graph class and then show that the proposed class is precisely the intersection of two better-known graph classes—the i-triangulated graphs of Gallai (1962) and the strongly orderable graphs of Dragan (2000).

We begin by describing several graph classes in terms of orderings of their vertex sets, as is important when dealing with algorithmic applications. There are additional characterizations (and even alternative notions of elimination orderings) of these classes in [2]; also see [3, 5, 6]. These order-theoretic definitions are only included as background; the graph-theoretic characterization of "generalized strongly chordal graphs" that will be given in Proposition 1 can most easily be used as their definition in reading this paper. Indeed, everything before Proposition 1 can be considered to be for background purposes only.

A graph is *chordal* if every cycle of length 4 or more has a chord. It has been observed many times since the 1960s (see [2]) that a graph G is chordal if and only if its vertex set $V(G)$ can be ordered by a *perfect elimination*

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ordering (this is the terminology in [2, 3]; it is called a *simplicial ordering* in [5]), which is defined to be any ordering that satisfies condition $\mathcal{C}1$, where

$\boldsymbol{\mathcal{C}1}$: If $a < b$ and $a < c$ with $ab, ac \in E(G)$, then $bc \in E(G)$.

As in Farber [6], a graph G is *strongly chordal* if $V(G)$ can be ordered by a *strong perfect elimination ordering* (in [2, 3, 6], or a *strong simplicial ordering* in [5]), which is defined to be any ordering that satisfies both conditions $\mathcal{C}1$ and $\mathcal{C}2$, where

$\boldsymbol{\mathcal{C}2}$: If $a < d$ and $b < c$ with $ab, ac, bd \in E(G)$, then $cd \in E(G)$.

See [2] for much more information on both chordal and strongly chordal graphs, including many of their standard graph-theoretic characterizations.

In [3], Dahlhaus defined a graph G to be *generalized strongly chordal* (now always called *strongly orderable*, starting with Dragan in [5]) if $V(G)$ can be ordered by a *generalized strong perfect elimination ordering* (in [3], called a *strong ordering* in [2, 5]), which is defined to be any ordering that satisfies condition $\mathcal{C}2$.

Dahlhaus proceeded to strengthen (or, as he wrote, to "improve") the definition of generalized strongly chordal graphs by further assuming that the generalized strongly perfect elimination ordering can always be chosen to end at a preselected arbitrary vertex (as is the case for [strongly] perfect elimination orderings of [strongly] chordal graphs). He used the verb "improve" because his proposed strengthening of the class of generalized strongly chordal graphs still contains the class of strongly chordal graphs, and so still generalizes strongly chordal graphs. He called the resulting graphs the *good generalized strongly chordal graphs*—which we will abbreviate as the *GGSC graphs*—and proved the cycle-and-chord characterization that is rephrased below as Proposition 1. In it, a *$^{\geq}k$-cycle* is a cycle of length at least k, and a chord xy of an even cycle C is a *strong chord* (as in [6]; it is called an *odd chord* in [2]) of C if the two x-to-y subpaths of C have odd lengths. Note that in condition $\langle 1.1 \rangle$, v can be either an endpoint or an internal vertex of the subpath of C.

Proposition 1 (Dahlhaus) *A graph is GGSC if and only if both the following hold*:

⟨1.1⟩ *For every odd* $^{\geq}5$*-cycle* C *and every* $v \in V(C)$*, there is a chord* xy *of* C *such that* v *is a vertex of the even-length* x*-to-*y *subpath of* C.

⟨1.2⟩ *Every even* $^{\geq}6$*-cycle has a strong chord.*

Figure 1 shows the eight graphs that can be induced by the vertex set of a 5-cycle C. The three graphs in the top row are not GGSC graphs, since ⟨1.1⟩ is violated by C and the vertex labeled v. Check that the five graphs in the bottom row are GGSC graphs.

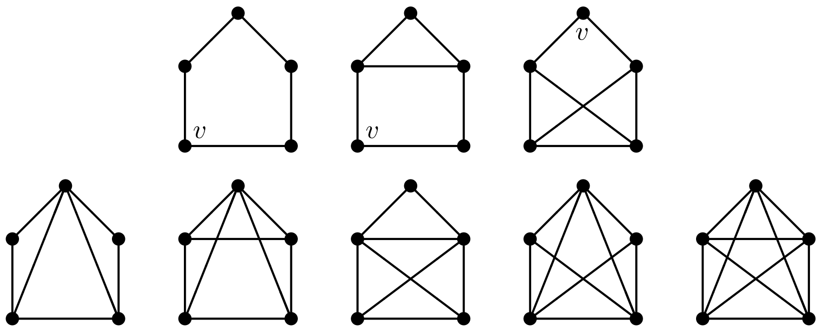

Figure 1: The graphs that can be induced by the vertices of a 5-cycle.

Farber [6] proved that a graph is strongly chordal if and only if it is chordal and every even $^{\geq}6$-cycle has a strong chord. Dahlhaus, Manuel, and Miller [4] proved that a graph is strongly chordal if and only if it is chordal and every $^{\geq}5$-cycle C has a 2*-chord triangle*, which is defined as a triangle formed by two chords of C and one edge of C. The following shows how GGSC graphs capture a bit of each of these.

Theorem 2 *A graph is GGSC if and only if both the following hold*:

⟨2.1⟩ *Every odd* $^{\geq}5$*-cycle has a* 2*-chord triangle.*

⟨2.2⟩ *Every even* $^{\geq}6$*-cycle has a strong chord.*

Proof. For the "only if" direction, suppose G is GGSC, arguing by induction on $n \geq 5$ for n-cycles C. We can assume that n is odd (otherwise C already has a strong chord by Proposition 1). The $n = 5$ basis case follows by checking that all $^{\geq}5$-cycles of the five graphs in the bottom row of Figure 1—the five GGSC graphs—have 2-chord triangles.

For the $n \geq 7$ induction step, Proposition 1 ensures that C has a chord xy. Suppose xy combines with the two x-to-y subpaths of C to form a k_1-cycle C_1 and a k_2-cycle C_2 where $k_1 \leq k_2$ and $k_1 + k_2 = n + 2$, and assume that xy has been chosen so as to make k_1 as small as possible, and so C_1 chordless. Thus, G being GGSC ensures $k_1 \not\geq 5$, and so $k_1 \in \{3, 4\}$.

Suppose for the moment that $k_1 = 4$ with (say) $\pi = x, x', y', y$ a subpath of C_1. Since n is odd, $k_2 = n - 2 \geq 5$ is odd, and so C_2 has a 2-chord triangle Δ by the induction hypothesis. We can assume that $xy \in E(\Delta)$ (otherwise Δ already is a 2-chord triangle of C); say $\Delta = wxy$ where wx and wy are chords of C_2. We can assume that w is not adjacent to either x' or y' (otherwise wxx' or wyy' already is a 2-chord triangle of C). Since C_1 is chordless, the 5-cycle formed from the path π and the edges wx and wy would have the unique chord xy, contradicting that G is GGSC.

Hence we can assume that $k_1 = 3$, say with C_1 the triangle xyz. Since n is odd, $k_2 = n - 1 \geq 6$ is even, and so C_2 has a strong chord ad by the inductive hypothesis with the four vertices x, y, a, d coming in that order around C_2 (possibly with $a = y$ or $d = x$). By repeatedly applying this argument to the even cycle formed by the edge ad and the a-to-d subpath of $C - z$, we can further assume that the chord ad has been chosen so as to partition C into two a-to-d subpaths as follows: a subpath $\pi = a, b, c, d$ of C_2 and a subpath τ of C through z. We can now assume that a and c are not adjacent and that b and d are not adjacent (otherwise acd or abd already is a 2-chord triangle of C). Let C' be the odd $^{\geq}5$-cycle formed by the path τ and the edge ad. Since C' has a 2-chord triangle Δ by the induction hypothesis, we can assume that $ad \in E(\Delta)$ (otherwise Δ already is a 2-chord triangle of C); say $\Delta = adw$ with $w \in V(\tau)$. Since G is GGSC, checking the five GGSC graphs in the bottom row of Figure 1 for the 5-cycle formed from the path π and the edges aw and dw (actually, only the second of these five has a candidate) shows that the 5-cycle must have chords bw and cw, and so bcw will be a 2-chord triangle of C.

For the "if" direction, suppose G satisfies conditions $\langle 2.1 \rangle$ and $\langle 2.2 \rangle$ and C is an n-cycle of G with $n \geq 5$ (toward showing that $\langle 1.1 \rangle$ and $\langle 1.2 \rangle$ hold). Since $\langle 2.2 \rangle$ ensures that $\langle 1.2 \rangle$ holds when n is even, we can assume that n is odd. By $\langle 2.1 \rangle$, C has a 2-chord triangle abw with chords aw and bw of C and $ab \in E(C)$. Thus, every $v \in V(C)$ is in both an a-to-w and a b-to-w subpath of C, one of which will have even length. Therefore, $\langle 1.1 \rangle$ also holds, and so G is a GGSC graph by Proposition 1. □

Two chords ab and cd of a cycle C are called *crossing chords* if their endpoints are distinct and come in the order a, c, b, d around C; otherwise, they are *noncrossing* chords. The *i-triangulated graphs* were introduced by Gallai in [7], where they were characterized by every odd $^{\geq}5$-cycle having noncrossing chords; also see [1, 8]. Recall that the *strongly orderable graphs* were introduced by Dragan in [5] to be exactly the same as the generalized strongly chordal graphs from [3]. Reference [2] contains further information on both of these graph classes, and Theorem 3 shows how GGSC graphs combine them.

Theorem 3 *A graph is GGSC if and only if it is both i-triangulated and strongly orderable.*

Proof. First suppose G is a GGSC graph. By Theorem 2, every odd $^{\geq}5$-cycle C of G has a 2-chord triangle, so C has two noncrossing chords, and so G is an i-triangulated graph by the characterization from [7] mentioned above. Since every GGSC graph is generalized strongly chordal as in [2], G is a strongly orderable graph by [5].

Conversely, suppose G is i-triangulated and strongly orderable, and C is an arbitrary odd $^{\geq}5$-cycle of G with $v \in V(C)$. Since G is i-triangulated, [1, Thm. 16.10] (or [8, Thm. 3]) ensures that, for every edge vw of C, there is a vertex z of C that forms a triangle with vw. Thus, for every vertex v of C, there is a chord xy of C (namely, vz or wz) such that v is a vertex of the even-length x-to-y subpath of C, and so G satisfies $\langle 1.1 \rangle$. Since G is strongly orderable, [3, Thm. 1] ensures that every even $^{\geq}6$-cycle of G has a strong chord, and so G satisfies $\langle 1.2 \rangle$. Therefore, by Proposition 1, G is a GGSC graph. □

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