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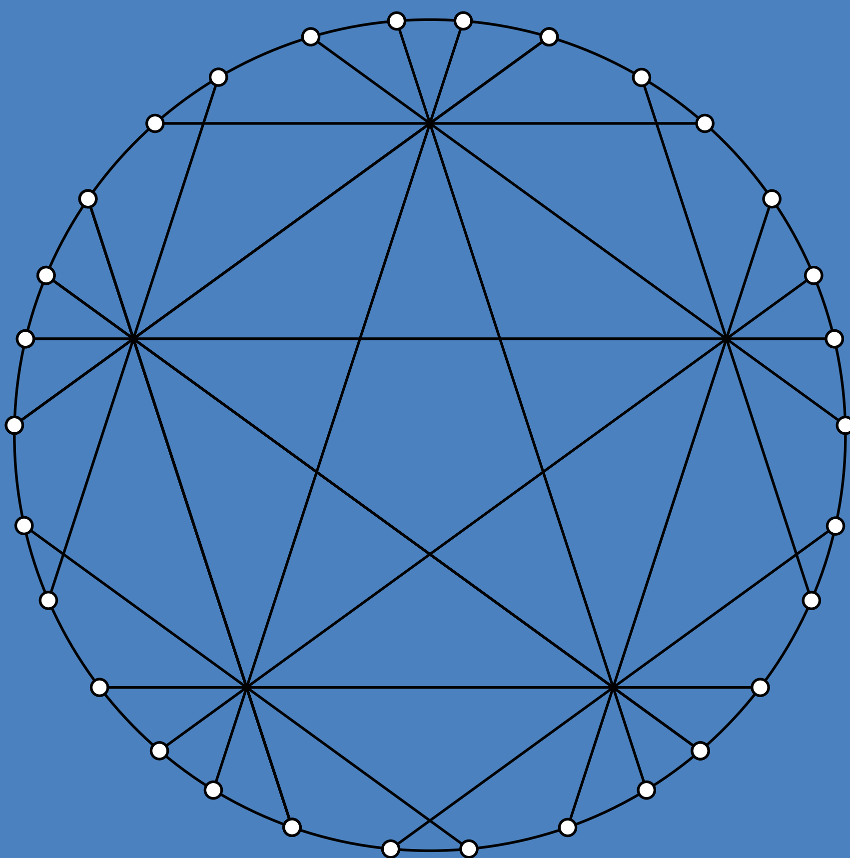

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On a Decomposition of a Non-simple 5-design Into 3-designs

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Abstract

We present a decomposition of the 5-$(12,6,2)$ design obtained from two identical copies of an $S(5,6,12)$ into twelve 3-$(12,6,2)$ designs.

1 Introduction

Let $\mathcal{B} = \{B_1, B_2, ..., B_b\}$ be a finite family of k-subsets (called *blocks*) of a point set $X = X(v) = \{1, 2, ..., v\}$. Then $(X, \mathcal{B})$ is a t-(v, k, λ) *design* if every t-subset of X is contained in exactly λ blocks of $\mathcal{B}$. A design without repeated blocks is called *simple*. Frequently, the point set X is implicit and we think of the design as just being the collection $\mathcal{B}$ of blocks. The set of all k-subsets of X will be denoted by $X^{(k)}$.

Let S_X denote the symmetric group on the symbols of X. For $\gamma \in S_X$,

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$x \in X$, $B \in X^{(k)}$ and $\mathcal{B} \subseteq 2^X$, we denote by $\gamma(x)$, $\gamma(B)$ and $\gamma(\mathcal{B})$ the images under γ of x, B and $\mathcal{B}$, respectively.

Let $(X, \mathcal{B})$ be a t-(v, k, λ) design. Then, an element $\gamma \in S_X$ is said to be an *automorphism* of the design if and only if $\gamma(\mathcal{B}) = \mathcal{B}$, that is, if and only if $\gamma(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$. The collection of all automorphisms of $(X, \mathcal{B})$ forms a subgroup of S_X called the *full automorphism group* of the design, and is denoted by $Aut(\mathcal{B})$. This group acts as a permutation group on the points and separately on the blocks of $\mathcal{B}$. Any subgroup H of $Aut(\mathcal{B})$ is simply called an automorphism group of the design.

A t-$(v, k, 1)$ design is called a *Steiner system* and the notation $S(t, k, v)$ is used in this case. An $S(3, 4, v)$ Steiner System $(X, \mathcal{B})$ is called a *Steiner Quadruple System* and is denoted by $SQS(v)(X, \mathcal{B})$. An $SQS(v)(X, \mathcal{B})$ is said to be (t, λ)-*resolvable* if its block-set $\mathcal{B}$ can be partitioned into r parts $\pi_1, \pi_2, ..., \pi_r$, $r \geq 2$, such that (X, π_i) is a t-$(v, 4, \lambda)$ design for all i. Clearly, $t = 1$ or 2. A (t, λ)-resolvable $SQS(v)$ is denoted by $RSQS(t, \lambda, v)$. The definition of a (t, λ)-resolvable SQS naturally extends to the definition of a (t, λ)-*resolvable* t'-(v, k, λ') *design*; here each part π_i must be a t-(v, k, λ) design with $t < t'$. The collection $\pi_1, \pi_2, ..., \pi_r$ is the (t, λ)-*resolution.*

2 Known Results

Resolutions of designs have been extensively studied in the case of a resolvable design, which is basically a resolution of a design into 1-designs [1], and in the case of large sets, which are resolutions of the trivial design $X^{(k)}$ [9]. Resolutions have applications in constructing other designs, coverings and packings. There are not many results on resolution of non-trivial t'-designs into t-designs with $t \geq 2$. In fact, most of the known results are on resolving $S(3, 4, v)$ into 2-$(v, 4, \lambda)$ designs. We are also aware of a couple of $(2, \lambda)$-resolution of 3-designs originating from codes and of two cases of a $(3, \lambda)$-resolution of a 5-design, one of which originates from a code as well.

Zaitsev et al. [13] proved the existence of an $RSQS(2, 1, 4^n)$:

Theorem 2.1 *There exists a* 3-$(4^n, 4, 1)$ *design that can be decomposed into* $\frac{4^n-2}{2}$ *disjoint* 2-$(4^n, 4, 1)$ *designs for all* $n \geq 2$.

The following recursive construction is due to Tierlinck [12].

Theorem 2.2 *If $k-1$ is a prime power, $k \equiv 8 \pmod{12}$, and an $RSQS(2,1,2k)$ exists, then an $RSQS(2,1,2(k-1)^n+2)$ exists for all $n \geq 1$.*

The last result can be combined with Theorem 2.1 to produce other infinite classes of $RSQS(2,1,v)$'s. The smallest value of v, $v \neq 4^n$ for which a $(2,1)$-resolvable $SQS(v)$ can be obtained by Theorem 2.2 is $v = 100$. Hartman [8] has found several $(2,3)$-resolvable SQS:

Theorem 2.3 *An $RSQS(2,3,v)$ exists for $v \in \{20, 32, 44, 68, 80, 104, 128\}$.*

We are only aware of two cases of a $(2,2)$-resolvable 3-designs, which are not SQS. Assmus and Salwach [3] showed that the weight 6 codewords of the extended binary $(16,11)$ Hamming code can be partitioned into 28 2-$(16,6,2)$ designs. They did not mention the codewords of weight 6 form a 3-$(16,6,16)$ design, but this is well-known, so the $(2,2)$-resolution of the 3-design follows. Another result of similar nature is implicitly present in [11]: The 3-$(16,6,4)$ design obtained from the weight 6 codewords in the extended Preparata code is $(2,2)$-resolvable. An algorithmic solution was given in [7], which shows the designs of the resolution can be extracted greedily, one by one, from the 3-$(16,6,4)$ design. We give a short alternative description here: Start with the 2-$(16,6,2)$ design

1	2	3	4	5	10	2	5	6	7	13	16
1	2	11	12	15	16	2	8	9	10	13	15
1	3	7	8	11	13	3	4	13	14	15	16
1	4	6	9	12	13	3	5	6	8	12	15
1	5	8	9	14	16	3	7	9	10	12	16
1	6	7	10	14	15	4	5	7	9	11	15
2	3	6	9	11	14	4	6	8	10	11	16
2	4	7	8	12	14	5	10	11	12	13	14

(one of the three non-isomorphic biplanes of order 4). If D_0 is the above design and f is the permutation $(1\ 9\ 13\ 5\ 14\ 8\ 3\ 16\ 12\ 10\ 2\ 15\ 11\ 6)(4\ 7)$, then the desired resolution is given by $D_0, D_1, \ldots, D_6$, where $D_i = f^i(D_0)$, $i = 1, 2, ..., 6$. The union of the designs $D_0, D_1, \ldots, D_6$ is a 3-$(16,6,4)$ design D with maximum intersection number 3. Another construction of D is based on the fact that one of the other two biplanes of order 4 (non-isomorphic to the one given above) contains "genetic" information about D [7].

Finally, we are aware of two results on a $(3, \lambda)$-resolution of a 5-design. The first one is not explicitly mentioned, but easily follows from the work presented in [4]; it is the resolution of a 5-$(48, 12, 8)$ into two 3-$(48, 12, 110)$ designs. The second is the decomposition of an $S(5, 6, 84)$ into 18 3-$(84, 6, 60)$ designs, and its double – the decomposition of two disjoint copies of an $S(5, 6, 84)$ into 36 3-$(84, 6, 60)$ designs [6].

In this article we show that the 5-$(12, 6, 2)$ design obtained from two identical copies of an $S(5, 6, 12)$ is $(3, 2)$-resolvable.

3 Some useful results and constructions

We list some results that will be used in the proof of our main result. The following extension theorem is due to Alltop [2].

Theorem 3.1 *Let $X = X(2k+1)$ and $D = (x, \mathcal{B})$ be a t-$(2k+1, k, \lambda)$ design with t even. Then*

$$\{B' : B' = X \backslash B,\ B \in \mathcal{B}\} \cup \{B'' : B'' = B \cup \{2k+2\}, B \in \mathcal{B}\}$$

is a $(t+1)$-$(2k+2, k+1, \lambda)$ design on the point set $X(2k+2)$.

Let HD denote a 3-$(12, 6, 2)$ design (known as Hadamard design, as it can be constructed from a Hadamard matrix of order 12). This design is unique and so is the Steiner System $S(5, 6, 12)$ [5]; both are well-known and well-studied structures. We list three properties of these two designs needed for proving our result in the next theorem. More properties and proofs can be found in [5].

Theorem 3.2 *([5])*

1. *Both the $S(5, 6, 12)$ and the HD are self-complementary, that is, whenever B is a block of the design, $X(12) \setminus B$ is also a block.*
2. *The maximum intersection of blocks of HD is 3. More precisely, for every block B of the HD, there is exactly one block of the HD disjoint from B and 20 other blocks each having intersection 3 with B.*
3. *There are exactly 12 HDs residing in an $S(5, 6, 12)$.*

We start by giving a simple description of the Steiner System $S(5,6,12)$, which, to our knowledge, has not been published before. There are other constructions known, including one from Hadamard matrix of order 12, and from an HD; apparently, these are all interrelated. A standard construction of an HD is to start with the symmetric 2-$(11,5,2)$ design given by developing the base block 1 2 3 7 10 with the automorphism (1 2 ... 11) and then extend it to an HD on $X(12)$ via the Alltop's construction given in Theorem 3.1. We denote this particular design by D^*. The $S(5,6,12)$ can then be obtained by the following.

Theorem 3.3 *Let $D = (X, \mathcal{B})$, $X = X(12)$, be a* 3-$(12,6,2)$ *design and let*

$$\mathcal{B}_D = \{A \in X^{(6)} : |A \cap B| \neq 5 \ \forall B \in \mathcal{B}\}.$$

Then $(X, \mathcal{B}_D)$ is an $S(5,6,12)$.

Proof. There are 22 6-sets of $X^{(6)}$ such that each intersects a block of D in 6 points (these 6-sets are the blocks of D). Now we count the number of 6-sets each intersecting a block of D in 5 points. No such set can intersect two or more blocks of D, as the maximum intersection number of two blocks of D is 3. There are 22 ways to choose a block B of D, $\binom{6}{5} = 6$ ways to choose a 5-subset of it, and 6 ways to choose the sixth point outside of B, for a total of $22(6)(6) = 792$ ways to choose a 6-set that intersects a block of D in 5 points. There are $\binom{12}{6} - 22 - 792 = 110$ 6-sets each intersecting any block of D in at most 4 points. Now, we know D resides in an $S(5,6,12)$. None of the 792 blocks can be a block of such $S(5,6,12)$, because if B' is such block, then there must be a block B'' of D, such that $|B' \cap B''| = 5$, and that would mean there is a 5-set covered by two different blocks of the $S(5,6,12)$, a contradiction. Hence the 110 blocks plus the blocks of D must be all the blocks of the $S(5,6,12)$ in which D resides, because D has 22 blocks and an $S(5,6,12)$ must have 132 blocks. ■

4 Main Result

Theorem 4.1 *The* 5-$(12,6,2)$ *design obtained from two identical copies of an $S(5,6,12)$ is* $(3,2)$*-resolvable.*

Proof. Let $X = X(12)$ and D_0 be the 3-$(12,6,2)$ design D^*. Set $D = D_0$ and let $(X(12), \mathcal{B}_{D_0})$ be the 5-$(12,6,1)$ design constructed in Theorem 3.3.

We need the following two permutations on $X(12)$ to describe the construction:

$$f = (5\ 11\ 10\ 9)(6\ 7\ 12\ 8) \text{ and } g = (1\ 4\ 3\ 2\ 10\ 6\ 9\ 11\ 7\ 5\ 8).$$

Let $D_1 = f(D_0)$, and $D_i = g^{i-1}(D_1)$, $i = 2, ...11$. We will show that $D_0, D_1, \ldots, D_{11}$ form the desired resolution. Both f and g are automorphisms of the 5-$(12, 6, 1)$ design $(X(12), \mathcal{B}_{D_0})$. We note that g is also an automorphism of D_0 and an automorphism of the entire set of the 12 designs $D_0, D_1, \ldots, D_{11}$; it fixes D_0 and rotates $D_1, D_2, \ldots, D_{11}$ cyclically ($g(D_{11}) = D_1$). To finish the proof, we observe that every two of the 12 designs intersect in exactly 2 complementary blocks. This follows from the fact that both f and g (and its powers) fix two complementary blocks. In other words, if p is any of f or g^i, $i = 1, 2, ..., 10$, D is an HD, and $D' = p(D)$ then the intersection of D' and D is two blocks complementary to each other. The union of the blocks of the 12 designs covers exactly 264 blocks, and all these blocks are blocks of the $S(5, 6, 12)$. Also, if we start with the blocks of one of the 12 designs, then add the blocks of a second one and so on, then the total number of new blocks added is $22 + 20 + ... + 2 + 0 = 132$, because of the mentioned intersection property. Likewise, the total number of repeated blocks added is $0 + 2 + ... + 20 + 22 = 132$, which shows that the union of all blocks of the 12 HDs is exactly two identical copies of the $S(5, 6, 12)$ in which the 12 HDs reside. ■

We can similarly obtain another $S(5, 6, 12)$, disjoint from the one obtained in Theorem 3.3, by using the same construction but starting from the 2-$(11, 5, 2)$ design obtained by developing the base block 1 2 3 5 8 with the automorphism (1 2 ... 11). If we denote this $S(5, 6, 12)$ by S_2 and the one obtained in Theorem 3.3 by S_1, then we can double S_2 and obtain similar decomposition into 12 HDs, each two of which intersect in exactly two complimentary blocks. Clearly, every HD residing in S_2 will be disjoint from every HD residing in S_1. It is known that two is the maximum number of disjoint $S(5, 6, 12)$ [10].

5 Conclusion

We have shown that the 5-$(12, 6, 2)$ design obtained from two identical copies of an $S(5, 6, 12)$ is $(3, 2)$-resolvable, and so is the 5-$(12, 6, 4)$ design obtained by two identical copies of the simple design $S_1 \cup S_2$. Although the designs we resolve are not simple, the resolutions are into simple designs.

Up to our knowledge, these are the first known $(3, \lambda)$-resolutions with minimum lambda.

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