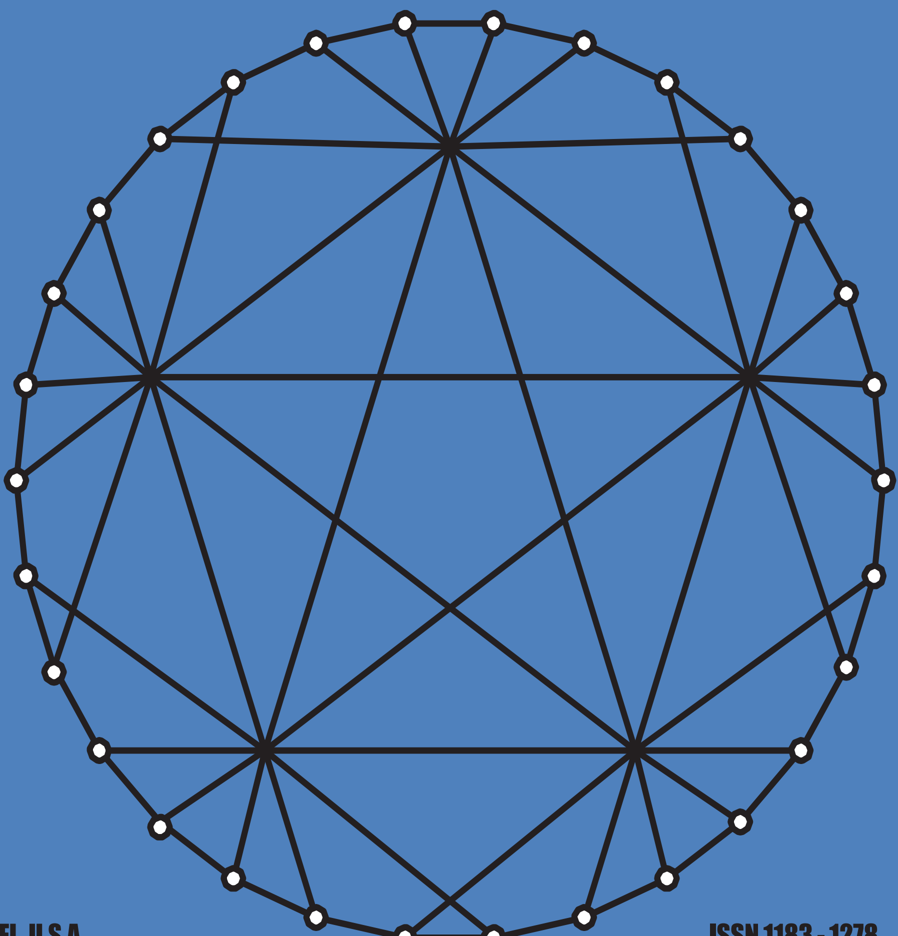
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 79
January 2017



Boca Raton, FL, U.S.A.

ISSN 1183 - 1278

BULLETIN OF THE ICA
Volume **79** (2017), Pages 95–107

Snakes: From Graceful to Harmonious

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Abstract

A vertex labeling is an assignment of integers to the vertices of a graph, subject to certain conditions. Graceful and harmonious labelings are among the most studied labelings. Suppose that G is a graph of order m and size n. The graph G is said to be *graceful* if there is an injection $f : V(G) \to \{0, 1, \ldots, n\}$ such that, when each edge uv of G has assigned the *weight* $|f(u) - f(v)|$, the resulting weights are distinct [14]. Similarly, when $m \leq n$, the graph G is said to be *harmonious* if there is an injection $f : V(G) \to \mathbb{Z}_n$ such that, when each edge uv of G has assigned the weight $f(u)+f(v) \pmod n$, the resulting weights are distinct [10].

In this paper we present a method that allows us to transform a special kind of graceful labeling into a harmonious labeling for four families of graphs that are constructed using cells isomorphic to the cycle C_4; thus we prove that all quadrilateral snakes, all snake polyominoes, all hybrid quadrilateral snakes, and all straight simple polyominal caterpillars are harmonious graphs.

1 Introduction

Let f be a graceful labeling of a graph G; suppose there exists an integer λ, such that $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$, for every $uv \in E(G)$, then f is said to be an *α-labeling* with *boundary value* λ. By an *α-graph* we mean a graph that admits an α-labeling. Note that an α-graph is necessarily bipartite.

A *snake* of length $n > 1$, is a packing of n congruent geometrical objects, called *cells*, such that the first and last cell each have only one neighbor and all the $n-2$ cells in between have exactly two neighbors. *Polyominoes* are planar shapes made by connecting a certain number of equal-sized squares, each joined together with at least one other square along an edge. A *snake polyomino* is a snake where the cells are squares, i.e., the cycle C_4. In [3], Barrientos and Minion proved that all snake polyominoes admit an α-labeling. A *nC_m-snake* is a connected graph in which the n cells (or blocks) are isomorphic to the cycle C_m and the block-cut point graph is a path. Thus, a nC_4-snake or *quadrilateral snake* is a bipartite graph of order $3n+1$ and size $4n$. Barrientos [1] proved that all quadrilateral snakes are α-graphs. A *hybrid quadrilateral snake* is a snake where two consecutive squares share a vertex or an edge.

Grace [8] defined a *sequential* labeling of a graph G of size n as an injective function $f : V(G) \to \mathbb{Z}_n$, where the *weight* of every edge uv of G is defined as $f(u) + f(v)$, and the set of induced weights is $\{t, t+1, \ldots, t+n-1\}$ for some integer t. A graph G is *sequential* if it admits a sequential labeling. Chang et al., [4], called the sequential labeling, *strongly t-harmonious.* Grace showed that any sequential graph is harmonious; he also proved the following proposition.

Proposition 1. *If G is an α-graph, then G is sequential.*

Motivated by all these results we investigate here the existence of α-labelings of snake polyominoes, quadrilateral snakes, hybrid quadrilateral snakes, that is, snakes where the connection is done via vertex amalgamation and/or edge amalgamation. We also consider here straight simple polyominal caterpillars; we present a technique that transforms the α-labelings of these graphs into harmonious labelings, proving so that all these graphs are harmonious.

The families of snakes considered in this work are quite robust; it is an open problem to determine the number of these snakes with n cells. There is an injection between the set of snake polyominoes with n cells and the set of self-avoiding walks of length $n-1$. For every snake polyomino, there exists a quadrilateral snake of the same length. The converse of this statement is not true. To check this fact, consider the examples in Figure 1.

In part (a) we have, in solid lines, a snake polyomino together with its associated quadrilateral snake (dashed lines). In part (b) we have, in solid lines, a quadrilateral snake, but the shape in dashed lines is not a valid snake polyomino. In part (c) we show that every snake polyomino can be represented, for enumeration purposes, as a self-avoiding walk. Note that not all self-avoiding walks correspond to a snake polyomino.

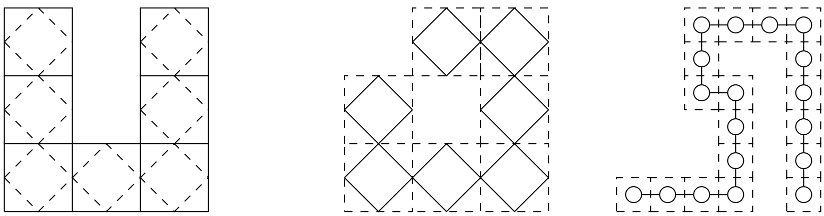

Figure 1: Relation between snake polyominos, quadrilateral snakes, and self-avoiding walks.

In this work we follow the notation and terminology used in [7]; the reader interested in graph labelings is refered to this survey for more information.

2 Related Results

Several families of harmonious graphs are known. Some of them are obtained using vertex amalgamation. For example, Seoud and Youseff [15] showed that if the one-point union of two cycles is harmonious, then the size of the graph is divisible by 4. Xu [16] proved that kC_3-snakes are harmonious if and only if $k \not\equiv 2(\text{mod } 4)$. Grace [9] showed that K_n-snakes are harmonious. Graham and Sloane [10] conjectured that $2K_n$-snake is harmonious only when n is 4.

We can also find in the literature harmonious labelings of graphs obtained using edge amalgamation. Graham and Sloane [10] proved that ladders, $P_m \times P_2$, are harmonious when $m > 2$; ladders are a special type of the snake polyominoes considered in this work. Jungreis and Reid [11] generalized this result by showing that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. Lu [13] used $\Theta(C_m)^n$ to denote the graph made from n copies of C_m that share an edge, that is, a generalized book. Xu [17] proved that $\Theta(C_m)^2$ is harmonious except when $m = 3$; moreover, he proved that all cycles with a chord are harmonious except for C_6 in the case where the distance in C_6 between the endpoints of the chord is 2.

In order to prove that the snakes considered in this work are harmonious, we prove first that they accept a felicitous labeling, these labelings were first used by Lee et al. in [12]. Let G be a graph of size n. An injective function $f : V(G) \to \mathbb{Z}_{n+1}$ is called *felicitous* if the weights induced by $f(u) + f(v) \pmod{n}$ for each edge uv of G are distinct. Figueroa-Centeno et al. [5] define a felicitous graph G to be *strongly felicitous* if there exists an integer λ so that for every edge uv of G, $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$. They prove that G is strongly felicitous if and only if G is an α-graph. For the sake of completeness, we prove here that every α-graph is also a felicitous graph.

Proposition 2. *If G is an α-graph, then G is felicitous.*

Proof. Suppose that G is an α-graph of size n. Let f be an α-labeling with boundary value λ of G. Consider the following labeling of G defined, for every vertex v of G, as

$$g(v) = \begin{cases} f(v) & \text{if } f(v) \leq \lambda, \\ n + \lambda + 1 - f(v) & \text{if } f(v) > \lambda. \end{cases}$$

Clearly, g is an injective function; furthermore, g uses labels from $[0, \lambda] \cup [\lambda + 1, n]$. For every $w \in \{1, 2, \ldots, n\}$, there is an edge uv of G such that $f(v) - f(u) = w$. The weight of the edge uv, under the labeling g is, $g(v)+g(u) = n+\lambda+1-f(v)+f(u) = n+\lambda+1-(f(v)-f(u)) = n+\lambda+1-w$. Since $1 \leq w \leq n$, we have that $\lambda + 1 \leq g(v) + g(u) \leq n + \lambda$. That is, the set of weights induced by g consists of n consecutive integers; when these integers are reduced modulo n, we obtain the set $\mathbb{Z}_n$. Therefore, g is a felicitous labeling of G. □

The results in Section 3 are based on three essential results related to α-labeled graphs. Let f be an α-labeling with boundary value λ of a graph G of size n, the labeling g of G defined as $g(v) = f(v)$ when $f(v) \leq \lambda$, and $g(v) = f(v) + k - 1$ when $f(v) > \lambda$, is a k-graceful labeling, that is, the set of induced weights is $\{k, k + 1, \ldots, k + n - 1\}$, where k is any positive integer. For $i = 1, 2$, let f_i be an α-labeling with boundary value λ_i of a graph G_i of size n_i. Suppose that f_1 is transformed into a $(n_2 + 1)$-graceful labeling and f_2 is shifted λ_1 units, by adding the constant λ_1 to every vertex label. Thus, both graphs have a vertex labeled λ_1. The amalgamation of these vertices produces a new α-labeled graph. Suppose now that f_1 is transformed into a n_2-graceful labeling and f_2 is shifted λ_1 units, both graphs have an edge of weight n_2 with end vertices labeled λ_1

and $\lambda_1 + n_2 - 1$. The edge amalgamation of these graphs results in a new α-labeled graph. For example, if the consecutive vertices of two copies of C_4 are labeled $0, 4, 2$, and 3, the boundary value is 2; the vertex and edge amalgamations produce the α-labeled graphs shown in Figure 2.

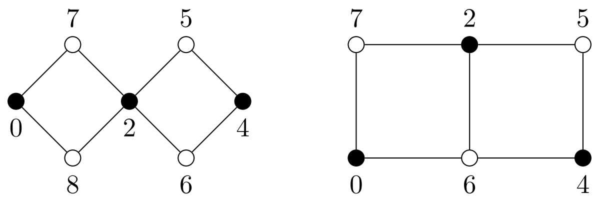

Figure 2: Vertex and edge amalgation of two squares.

3 Harmonious Snakes

The concept of chain graphs was introduced by Barrientos in [2]. The following definition extends this concept. For every $1 \leq i \leq n$, let $u_i, v_i \in V(G_i)$, where G_i is a connected graph; the graph G, obtained by identifying (vertex amalgamation) v_i with u_{i+1}, for each $1 \leq i \leq n-1$, is called a *chain graph*. Barrientos [2] proved that when all the G_i are α-graphs, there exists a chain graph G, constructed using the G_i's, that is also an α-graph. To prove this result, the vertices u_i and v_i are chosen to be the vertices labeled 0 and λ by an α-labeling of the G_i's.

So, we can prove that all quadrilateral snakes are α-graphs by showing that we have an α-labeling of C_4 that allows us to do the vertex amalgamation. Consider the labeling of $G_i = C_4$ that assigns the integers $x_i, y_i, x_i + 2$, and $y_i + 1$ to consecutive vertices. If $x_i = 0$ and $y_i = 3$, this is an α-labeling. If $x_i + 2 < y_i$, the induced weights are $y_i - x_i - 2, y_i - x_i - 1, y_i - x_i$, and $y_i - x_i + 1$; that is, four consecutive numbers. If $x_i > y_i + 1$, the weights are $x_i - y_i - 1, x_i - y_i, x_i - y_i, x_i - y_i + 1$, and $x_i - y_i + 2$. Note that this is an α-labeling of C_4 when $y_i = 0$ and $x_i = 2$. Suppose that u_i is labeled x_i when $x_i + 2 < y_i$, then v_i can be chosen to be the vertex labeled $x_i + 2$ or $y_i + 1$. If v_i is labeled $x_i + 2$, then u_{i+1} must be labeled $x_{i+1} = x_i + 2$ and $x_{i+1} + 2 < y_{i+1}$. If v_i is labeled $y_i + 1$, then u_{i+1} must be labeled $y_{i+1} = y_i + 1$. Thus, the weights on each G_i are four consecutive integers and the final labeling of G is an α-labeling.

Using this construction, Barrientos [1] proved the next theorem.
Theorem 3.1. *All quadrilateral snakes are α-graphs.*

As a consequence of this theorem and Proposition 2 we can prove the following corollary.

Corollary 1. *All quadrilateral snakes are felicitous.*

In Figure 3a we show the α-labeling, of a quadrilateral snake of length 7, obtained using this procedure. In Figure 3b we show the felicitous labeling of the snake obtained using Proposition 2.

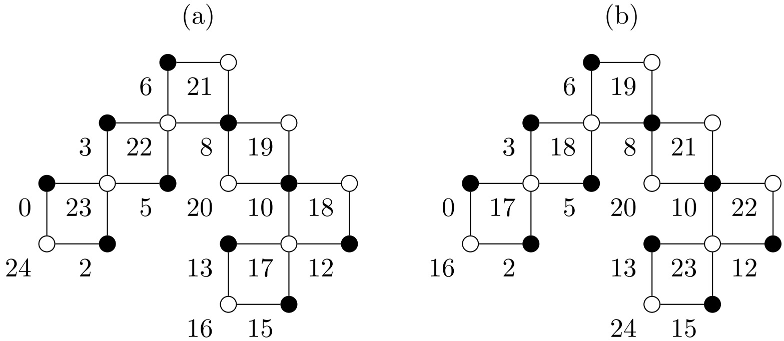

Figure 3: α- and felicitous labelings of a quadrilateral snake

Theorem 3.2. *All quadrilateral snakes are harmonious.*

Proof. Let G be a quadrilateral snake of length n, thus G is a graph of size $4n$. Let g be the felicitous labeling of G described before. If λ is the boundary value of the α-labeling f of G used to generate g, then the labels on the first cell of G are $0, \lambda+1, 2, \lambda+2$, with induced weights $\lambda+1, \lambda+2, \lambda+3$, and $\lambda+4$. The labels on the last cell of G are $\lambda-2, 4n, \lambda$, and $4n-1$, with induced weights $\lambda-3, \lambda-2, \lambda-1$, and λ. Now we replace the labels $4n$ and λ with the labels 1 and $\lambda-1$, respectively. Note that neither of these labels has been used before, so the new labeling of G is injective and uses labels from $\mathbb{Z}_{4n}$. The weights induced by this new labeling of G on the last cell are also $\lambda-3, \lambda-2, \lambda-1$, and λ. Since neither of the labels nor the weights, on the other cells, have been modified and the largest label used is $4n-1$, we have a harmonious labeling of G. □

In Figure 4 we show the resulting labeling for the graph in Figure 2.

Now we turn our attention to the snake polyominoes. We claim that all snake polyominoes of length $n > 1$ are harmonious. The technique used to prove this claim does not produce the desired labeling when the polyomino is the one in Figure 5, however this graph is harmonious, as we show in Figure 5.

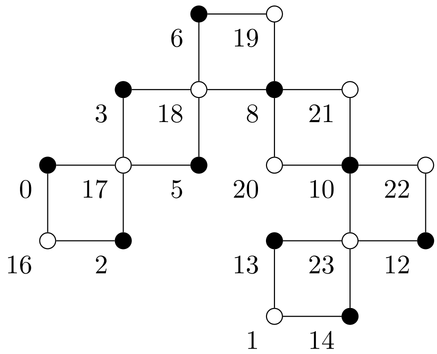

Figure 4: A harmonious labeling of a quadrilateral snake.

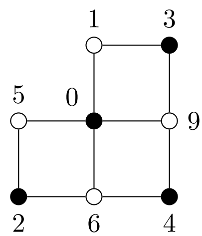

Figure 5: Harmonious labeling of a special snake polyomino of length 3.

Let G_1 and G_2 be two graphs of positive sized. The graph Γ obtained by identifying an edge of G_1 with an edge of G_2 is called an *edge amalgamation* of G_1 and G_2. In [3], Barrientos and Minion proved the following result.

Theorem 3.3. *If G_1 and G_2 are two α-graphs, then there is an edge amalgamation Γ of G_1 and G_2 that is an α-graph.*

The graph Γ is obtained by amalgamating the edge of weight 1 in G_2 with the edge of weight $|E(G_2)|$ in G_2.

Using this result, together with the building blocks shown in Figure 6, the authors were able to prove that all snake polyominoes admit an α-labeling [3].

We claim that all snake polyominoes are harmonious. In order to prove this claim, we first show an α-labeling of the snake that has a property that allows us to transform it into a felicitous labeling, which is finally transformed into the claimed harmonious labeling. This α-labeling is, in general, different of that found in [3].

Theorem 3.4. *All snake polyominoes are α-graphs.*

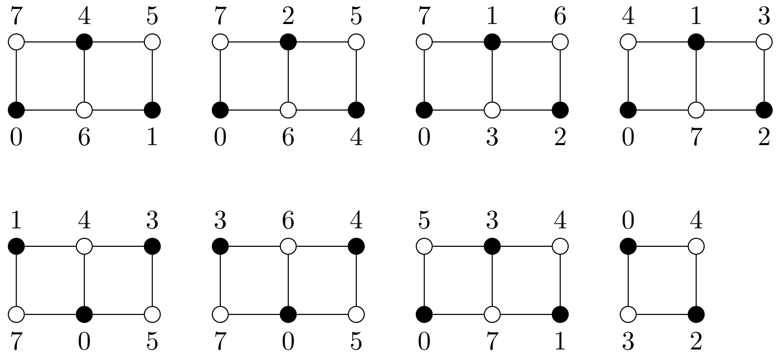

Figure 6: α-labelings of the building blocks.

Proof. Suppose that G is a snake polyomino with n cells. Then its order in $2(n+1)$ and size $3n+1$. We analyze two cases depending on the parity of n.

Case 1: When n is even. Suppose that G has been α-labeled using the technique presented in [3], Theorem 2. Thus, the weights in the first cell of G are $3n-2, 3n-1, 3n$, and $3n+1$; and the weights in the last cell of G are $4, 3, 2$, and 1. We decompose G into three components G_1, G_2, and G_3, where G_1 and G_3 are the extreme cells of G, respectively, and G_2 is the snake polyomino formed by all the interior cells of G. Note that when $n = 2$, G_2 does not exist. Substract from every vertex label of G_2 the constant that transforms the labeling of G_2 into an α-labeling. The components G_1 and G_2 are labeled using the α-labeling of C_4 shown in Figure 6. Thus, G_1, G_2, and G_3 have been α-labeled and the edge of G where G_1 and G_2 were amalgamated has weight $3n-5$; similarly, the edge of G where G_2 and G_3 were amalgamated has weight 1. Therefore, we can apply Theorem 3 to G_1 and G_2, to produce a snake polyomino, G', with $n-1$ cells, identifying the edge of weight 1 in G_1 with the edge of $3n-5$ in G_2. Once this is done, we apply Theorem 3 again to G' and G_3, identifying the edge of weight 1 in G' with the edge of weight 4 in G_3. The resulting graph is G with an α-labeling f that does not use the labels 1 and $\lambda - 1$, where λ is its boundary value.

Case 2: When n is odd. We decompose G into two components, G_1 and G_2, where G_1 consists of the first $n-3$ cells of G and G_2 is formed by the last three cells of G. Since $n-3$ is even, we label G_1 using the technique described in Case 1. G_2 is α-labeled using one of the labelings shown in Figure 7, the labeling is chosen in such a way that the edge of G, where G_1 and G_2 were amalgamated, has weight 10.

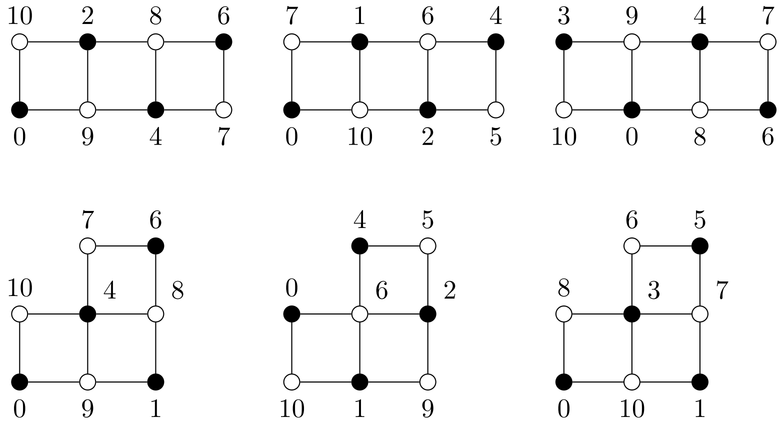

Figure 7: α-labelings of snake polyominoes with three cells.

It is routine to verify that these building blocks cover all the possible configurations of the last three cells of G.

Now we apply Theorem 3 to G_1 and G_2, identifying the edge of weight 1 in G_1 with the edge of weight 10 in G_2. Thus, as in Case 1, we have obtained an α-labeling of G that does use the labels 1 and $\lambda - 1$.

Therefore, any snake polyomino with n cells has an α-labeling that does not use the labels 1 and $\lambda - 1$. □

As a consequence of Proposition 2 and Theorem 4 we have the following corollary.

Corollary 2. *If G is a snake polyomino with $n \geq 2$ cells, then G is felicitous.*

As we did in Theorem 2, we use the felicitous labeling of G to prove that G is also a harmonious graph.

Theorem 3.5. *If G is a snake polyomino with $n \geq 2$ cells, then G is harmonious.*

Proof. The α-labeling of G obtained in Theorem 4 does not use the labels 1 and $\lambda - 1$. Thus, the associated felicitous labeling of G does not use these labels neither. So, to obtain a harmonious labeling we replace the label $3n + 1$ by 1 and the label λ by $\lambda - 1$. As in Theorem 2, the final labeling is in fact a harmonious labeling of G. □

In Figure 8 we show a complete example of this construction, starting with the α-labeling and ending with the associated harmonious labeling.

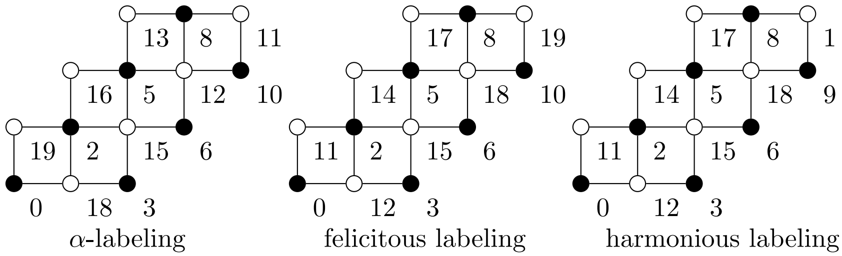

Figure 8: α-, felicitous, and harmonious labeling of a snake polyomino

Now we describe how to obtain an α-labeling of a hybrid quadrilateral snake. By a *hybrid quadrilateral snake* we mean a snake where every interior cell shares an edge or a vertex with its neighbor cells. In other words, the $(i+1)^{\text{th}}$ cell is attached to the i^{th} cell using vertex or edge amalgamation. Since every cell is an α-graph, these operations produce an α-graph when the building blocks are used according to the procedures described in Theorem 1 and Theorem 4. Thus, we can prove the following theorem.

Theorem 3.6. *All hybrid quadrilateral snakes are α-graphs.*

Using the building blocks in Figure 6 it is always possible to start with a cell that does not use the label 1 and end with a cell that does not use the label $\lambda - 1$. This implies that the resulting α-labeling, of the hybrid snake, can be transformed into a felicitous labeling, which can be modified to get a harmonious labeling of this snake. In Figure 8 we show a harmonious labeling of a hybrid snake with 9 cells obtained using the constructions described in this work.

Corollary 3. *All hybrid snakes are felicitous.*

Corollary 4. *All hybrid snakes are harmonious.*

Froncek et al. [6] introduced the concept of straight simple polyominal caterpillar as the graph obtained as follows. Consider two paths of length n, Π_1 and Π_2, with $V(\Pi_i) = \{u^i_1, u^i_2, \ldots, u^i_{n+1}\}$ and $E(\Pi_i) = \{u^i_j u^i_{j+1} : 1 \leq j \leq n\}$. With these paths a (straight) snake polyomino (or ladder $P_{n+1} \times P_2$) can be formed by adding the edges $u^1_j u^2_j$ for every $1 \leq j \leq n+1$. Let L be any nonempty subset of $\{u^1_j u^1_{j+1}, u^2_j u^2_{j+1} : 1 \leq j \leq n\}$. A *straight simple polyominal caterpillar* is the graph obtained by edge amalgamating a copy of C_4 to every element of L. In [6], the authors proved that all straight simple polyominal caterpillars are α-graphs. In order to prove that these graphs also admit a harmonious labeling, is enough to show that there exists an α-labeling that does not assign the labels 1 and $\lambda - 1$. The

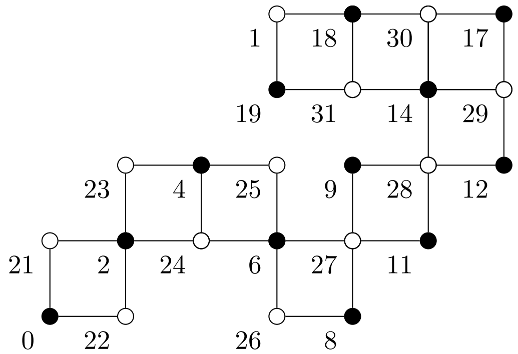

Figure 9: Harmonious labeling of a hybrid quadrilateral snake.

building blocks used in [6] not always produce an α-labeling with these characteristics. The labeled building blocks shown in Figure 10 solve this problem. The building blocks in parts (a) - (d) can be applied to label the first cells(s) (or head) of our graph without using the label 1; the building blocks in parts (c) - (f) can be applied to label the last cell(s) (or tail) of our graph without using the label $\lambda-1$. The cells in between can be labeled using these blocks or using the blocks in [6]. Note that the labelings in (e) and (f) are based on a specific coloring of the vertices, if this coloring does not match the one of the rest of the polyominal caterpillar, then it is always possible to relabel them in such a way that they match with the coloring of these two building blocks. Thus we have proved the following.

Proposition 3. *For any straight simple polyominal caterpillar, G, there exists an α-labeling of G with boundary value λ such that the integers 1 and $\lambda - 1$ are not assigned as labels of G.*

As we concluded before, this special α-labeling can be transformed into a felicitous labeling, which can be modified into a harmonious labeling. Thus, we can prove the following corollaries.

Corollary 5. *All straight simple polyominal caterpillars are felicitious.*

Corollary 6. *All straight simple polyominal caterpillars are harmonious.*

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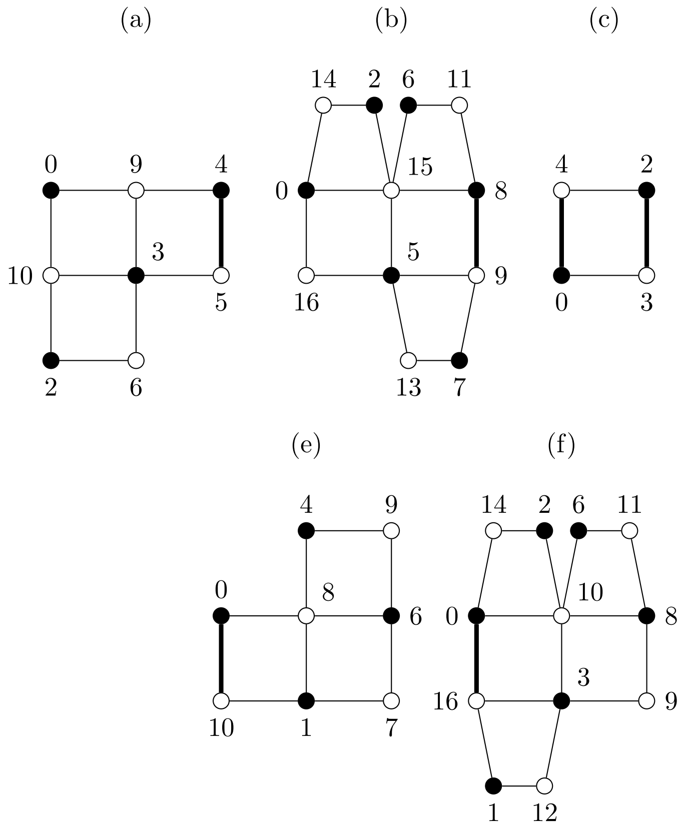

Figure 10: α-labeled building blocks.

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