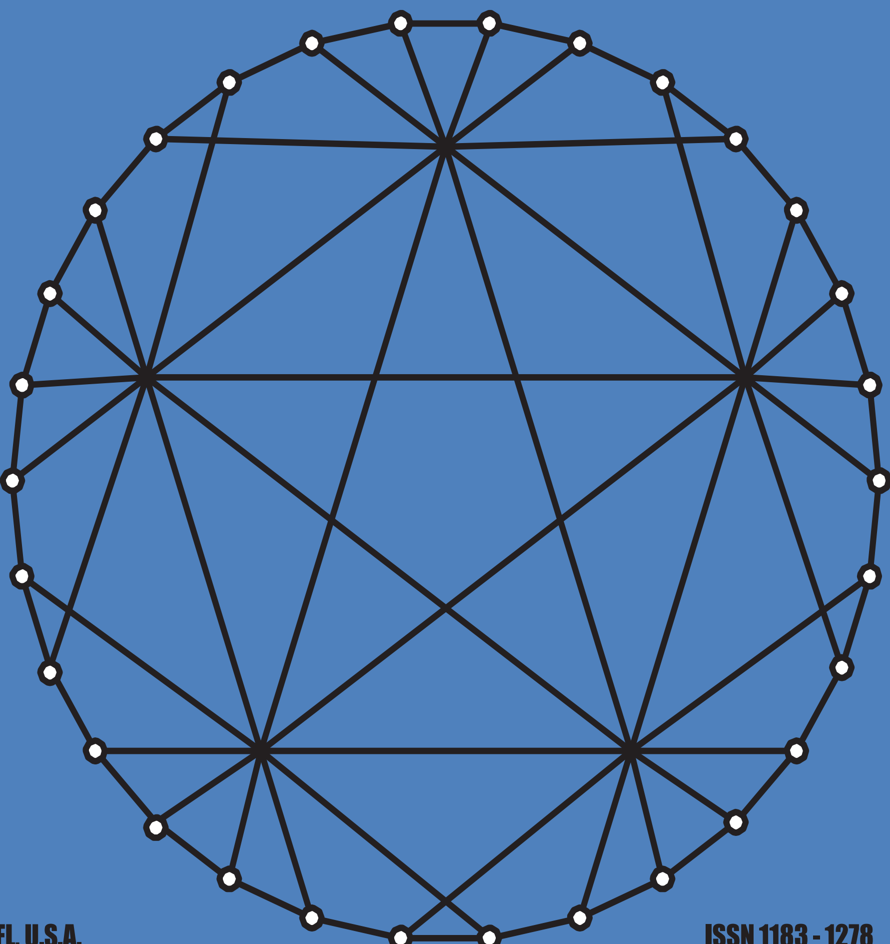
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 79
January 2017



Boca Raton, FL, U.S.A.

ISSN 1183 - 1278

BULLETIN OF THE ICA
Volume **79** (2017), Pages 82–94

On Sequence-Based Closed Form Entries for an Exponentiated General 2×2 Matrix: A Re-Formulation and an Application

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Abstract

Closed form entries for an exponentiated (and arbitrary) 2×2 matrix are established here, and expressed in terms of a specialized Horadam sequence; two proofs of the result are given accordingly, along with examples and observations derived therefrom. The result offers a new formulation of a general class of polynomial families associated with sequences whose ordinary generating functions are governed by quadratic equations.

1 Introduction and Result

1.1 Background

Let $\mathbf{M}$ be the general 2×2 matrix

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1}$$

with each of A, B, C, D assumed non-zero along with its trace $T(A, D) = A + D$ and determinant $M(A, B, C, D) = |\mathbf{M}| = AD - BC$. In a publication [2] it was shown that the anti-diagonals ratio B/C is independent of any power $n \geq 1$ to which the matrix $\mathbf{M}$ is raised; four proofs were offered accordingly. Closed form entries for the matrix $\mathbf{M}^n$, formulated by McLaughlin in 2004 [8, Theorem 1, p. 3], delivers this result trivially, and the article [3] has provided a new derivation of these entries using so called Catalan polynomials that are intimately connected to a fundamental parameter of McLaughlin. It has come to the attention of the authors that McLaughlin's result actually appears in predating work by Pla [9, Lemma 1, p. 440], who makes the same statement but couched in the language of the well known Horadam sequence; it is his version on which we focus here, initially giving two proofs and some illustrative examples. Immediate observations follow, together with an application in which members of previously studied polynomial families are—in any family instance—identified as being two (equivalent) instances of what may be termed generalized Fibonacci polynomials.

1.2 Result and Examples

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence arising from the general second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.2}$$

for which $w_0 = a$ and $w_1 = b$ are initial values. Then we have the following result:

Theorem 1.1. *For $n \geq 1$,*

$$\mathbf{M}^n(A, B, C, D) = \begin{pmatrix} w^*_{n+1} - Dw^*_n & Bw^*_n \\ Cw^*_n & w^*_{n+1} - Aw^*_n \end{pmatrix},$$

where w^*_n is the general term of the particular Horadam sequence $\{w_n(0, 1; T, M)\}_0^{\infty}$.

Pla asked if it is possible for an element of the matrix $\mathbf{M}^n$ to describe precisely a Horadam sequence term in its full generality, establishing the restricted case above as an affirmative one. Note that the part generalization $\{w_n(0,1;p,q)\}_0^\infty$ is, historically, known as the (fundamental) generalized Fibonacci sequence, of which $\{w_n(0,1;T,M)\}_0^\infty$ is one particular instance. The (primordial) generalized Lucas sequence $\{w_n(2,p;p,q)\}_0^\infty$ also features in Pla's work, both types of sequence being mentioned in the Horadam sequence survey article [7] to which the reader is directed if interested. Further background on closed form entries for exponentiated 2-square (and higher order) matrices is to be found in [3].

Example 1. Setting $A=2$, $B=-3$, $C=-5$ and $D=7$, Theorem 1.1 reads

$$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}^n = \begin{pmatrix} w_{n+1}^* - 7w_n^* & -3w_n^* \\ -5w_n^* & w_{n+1}^* - 2w_n^* \end{pmatrix}, \tag{1.3}$$

where $\{w_n^*\}_0^\infty = \{w_n(0,1;9,-1)\}_0^\infty = \{0,1,9,82,747,6805,\ldots\}$, correctly giving

$$\begin{aligned} \begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}^5 &= \begin{pmatrix} w_6^* - 7w_5^* & -3w_5^* \\ -5w_5^* & w_6^* - 2w_5^* \end{pmatrix} \\ &= \begin{pmatrix} 61992 - 7\cdot 6805 & -3\cdot 6805 \\ -5\cdot 6805 & 61992 - 2\cdot 6805 \end{pmatrix} \\ &= \begin{pmatrix} 14357 & -20415 \\ -34025 & 48382 \end{pmatrix}. \end{aligned} \tag{1.4}$$

Example 2. Setting $A=7$, $B=-\sqrt{5}$, $C=2\sqrt{2}$ and $D=3$, Theorem 1.1 reads

$$\begin{pmatrix} 7 & -\sqrt{5} \\ 2\sqrt{2} & 3 \end{pmatrix}^n = \begin{pmatrix} w_{n+1}^* - 3w_n^* & -\sqrt{5}w_n^* \\ 2\sqrt{2}w_n^* & w_{n+1}^* - 7w_n^* \end{pmatrix}, \tag{1.5}$$

where $\{w_n^*\}_0^\infty = \{w_n(0,1;10,21+2\sqrt{10})\}_0^\infty = \{0,1,10,79-2\sqrt{10},20(29-2\sqrt{10}),4181-516\sqrt{10},\ldots\}$, correctly giving

$$\begin{aligned} \begin{pmatrix} 7 & -\sqrt{5} \\ 2\sqrt{2} & 3 \end{pmatrix}^2 &= \begin{pmatrix} w_3^* - 3w_2^* & -\sqrt{5}w_2^* \\ 2\sqrt{2}w_2^* & w_3^* - 7w_2^* \end{pmatrix} \\ &= \begin{pmatrix} 49 - 2\sqrt{10} & -10\sqrt{5} \\ 20\sqrt{2} & 9 - 2\sqrt{10} \end{pmatrix}. \end{aligned} \tag{1.6}$$

The result has been tested exhaustively by computer using rational and complex entries also for the matrix $\mathbf{M}$.

2 Proofs, Observations and an Application

2.1 Two Proofs

We provide two different proofs of Theorem 1.1, and some observations accordingly.

Proof I

This proof is a straightforward inductive one.

Proof. Noting that (1.2) is satisfied by the sequence w_n^*, with $w_2^* = Tw_1^* - Mw_0^* = T(1) - M(0) = T$, we see that the result is valid for $n = 1$, correctly reading

$$\begin{aligned} \mathbf{M} \;=\; \mathbf{M}^1 &= \begin{pmatrix} w_2^* - Dw_1^* & Bw_1^* \\ Cw_1^* & w_2^* - Aw_1^* \end{pmatrix} \\ &= \begin{pmatrix} T - D & B \\ C & T - A \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned} \tag{I.1}$$

We assume the same is true for some $n = k \geq 1$, and thus consider

$$\begin{aligned} \mathbf{M}^{k+1} &= \mathbf{M}\mathbf{M}^k \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w_{k+1}^* - Dw_k^* & Bw_k^* \\ Cw_k^* & w_{k+1}^* - Aw_k^* \end{pmatrix} \\ &= \begin{pmatrix} Aw_{k+1}^* - Mw_k^* & Bw_{k+1}^* \\ Cw_{k+1}^* & Dw_{k+1}^* - Mw_k^* \end{pmatrix}. \end{aligned} \tag{I.2}$$

Now $-Mw_k^* = w_{k+2}^* - Tw_{k+1}^*$ by (1.2), so that

$$\begin{aligned} \mathbf{M}^{k+1} &= \begin{pmatrix} w_{k+2}^* - (T - A)w_{k+1}^* & Bw_{k+1}^* \\ Cw_{k+1}^* & w_{k+2}^* - (T - D)w_{k+1}^* \end{pmatrix} \\ &= \begin{pmatrix} w_{k+2}^* - Dw_{k+1}^* & Bw_{k+1}^* \\ Cw_{k+1}^* & w_{k+2}^* - Aw_{k+1}^* \end{pmatrix}, \end{aligned} \tag{I.3}$$

and the inductive step is upheld. □

As alluded to above, the result appears in [8] with entries of the exponentiated matrix $\mathbf{M}$ expressed in terms of a parameter

$$\begin{aligned} y_n = y_n(A,B,C,D) &= y_n(T(A,D),M(A,B,C,D)) \\ &= \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} T^{n-2i}(-M)^i, \end{aligned} \tag{2.1}$$

from which it is immediate that

$$\{y_n(T,M)\}_0^\infty = \{w_{n+1}^*\}_0^\infty = \{w_{n+1}(0,1;T,M)\}_0^\infty. \tag{2.2}$$

Proof II

This proof is a constructive one.

Proof. First, we state a subsidiary result, about which we make a remark.

Lemma 2.1. *For $n \geq 1$,*

$$\begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} w_{n+1}^\dagger & -qw_n^\dagger \\ w_n^\dagger & -qw_{n-1}^\dagger \end{pmatrix},$$

where $\{w_n^\dagger\}_0^\infty$ is the generalised Fibonacci sequence $\{w_n(0,1;p,q)\}_0^\infty$; we omit an inductive proof as elementary, utilising as it does merely (1.2).

Remark 2.1. Although Lemma 2.1 is deployed here for the proof of Theorem 1.1, we remark that it is in fact a special case of Theorem 1.1 on setting $A = p$, $B = -q$, $C = 1$ and $D = 0$ in the latter (obtaining the bottom right-hand entry $-qw_{n-1}^\dagger$ of Lemma 2.1 requires the use of (1.2)), noting that in this instance $\{w_n^*\}_0^\infty = \{w_n(0,1;T,M)\}_0^\infty = \{w_n(0,1;p,q)\}_0^\infty = \{w_n^\dagger\}_0^\infty$, from which the observation follows readily.

As for the proof itself, let us define a matrix

$$\mathbf{X}(A,B) = \begin{pmatrix} 0 & B \\ 1 & -A \end{pmatrix}, \tag{II.1}$$

from which it is found that

$$\mathbf{X}^{-1}(A,B)\mathbf{M}(A,B,C,D)\mathbf{X}(A,B) = \mathbf{S}(T,M), \tag{II.2}$$

say, where $\mathbf{S}(T,M)$ has form

$$\mathbf{S}(T,M) = \begin{pmatrix} T & -M \\ 1 & 0 \end{pmatrix} \tag{II.3}$$

and in turn, invoking Lemma 2.1,

$$\begin{aligned} \mathbf{S}^n(T,M) &= \begin{pmatrix} w_{n+1}(0,1;T,M) & -Mw_n(0,1;T,M) \\ w_n(0,1;T,M) & -Mw_{n-1}(0,1;T,M) \end{pmatrix} \\ &= \begin{pmatrix} w^*_{n+1} & -Mw^*_n \\ w^*_n & -Mw^*_{n-1} \end{pmatrix}. \end{aligned} \tag{II.4}$$

From (II.2), therefore, it follows that

$$\begin{aligned} &\mathbf{M}^n(A,B,C,D) \\ &\quad = \mathbf{X}(A,B)\mathbf{S}^n(T,M)\mathbf{X}^{-1}(A,B) \\ &\quad = \frac{1}{B}\begin{pmatrix} B(Aw^*_n - Mw^*_{n-1}) & B^2 w^*_n \\ Aw^*_{n+1} - (M+A^2)w^*_n + AMw^*_{n-1} & B(w^*_{n+1} - Aw^*_n) \end{pmatrix} \end{aligned} \tag{II.5}$$

by (II.1) and (II.4), after a little algebra. The elements of the second column of $\mathbf{M}^n$ are correct by inspection of Theorem 1.1, and those of the first are confirmed as such by a routine application of (1.2) (trivial reader exercise). □

An immediate consequence of Theorem 1.1 is the following corollary:

Corollary 2.1. *Any 2×2 matrix $\mathbf{M}$ having trace 1 and determinant -1 has entries involving combinations and multiples of Fibonacci numbers when exponentiated, with form*

$$\mathbf{M}^n = \begin{pmatrix} F_{n+1} - DF_n & BF_n \\ CF_n & F_{n+1} - AF_n \end{pmatrix}, \qquad n \geq 1.$$

Proof. We see that $\{w^*_n\}_0^\infty = \{w_n(0,1;1,-1)\}_0^\infty = \{0,1,1,2,3,5,\ldots\} = \{F_n\}_0^\infty$, the Fibonacci sequence. □

A couple of known examples of Corollary 2.1 are

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n+1} - F_n \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{2.3}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} - F_n & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}, \tag{2.4}$$

with Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ yielded on taking the determinant of l.h.s. and r.h.s. matrices in each result. A different example here is

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ 1 & \frac{1}{2} \end{pmatrix}^n = \begin{pmatrix} F_{n+1} - \frac{1}{2}F_n & \frac{5}{4}F_n \\ F_n & F_{n+1} - \frac{1}{2}F_n \end{pmatrix}, \tag{2.5}$$

which correctly returns, for instance,

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ 1 & \frac{1}{2} \end{pmatrix}^4 &= \begin{pmatrix} F_5 - \frac{1}{2}F_4 & \frac{5}{4}F_4 \\ F_4 & F_5 - \frac{1}{2}F_4 \end{pmatrix} = \begin{pmatrix} 5 - \frac{1}{2}\cdot 3 & \frac{5}{4}\cdot 3 \\ 3 & 5 - \frac{1}{2}\cdot 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{7}{2} & \frac{15}{4} \\ 3 & \frac{7}{2} \end{pmatrix}, \end{aligned} \tag{2.6}$$

and another illustration is

$$\begin{aligned} \begin{pmatrix} 2 & -\frac{1}{4} \\ 4 & -1 \end{pmatrix}^7 &= \begin{pmatrix} F_8 + F_7 & -\frac{1}{4}F_7 \\ 4F_7 & F_8 - 2F_7 \end{pmatrix} = \begin{pmatrix} 21 + 13 & -\frac{1}{4}\cdot 13 \\ 4\cdot 13 & 21 - 2\cdot 13 \end{pmatrix} \\ &= \begin{pmatrix} 34 & -\frac{13}{4} \\ 52 & -5 \end{pmatrix}. \end{aligned} \tag{2.7}$$

A further example, this time involving complex numbers, is

$$\begin{aligned} \begin{pmatrix} 1+i & 2-i \\ 1 & -i \end{pmatrix}^3 &= \begin{pmatrix} F_4 + F_3 i & F_3(2-i) \\ F_3 & F_4 - F_3(1+i) \end{pmatrix} \\ &= \begin{pmatrix} 3+2i & 2(2-i) \\ 2 & 1-2i \end{pmatrix}. \end{aligned} \tag{2.8}$$

A different special case provides an additional result.

Corollary 2.2. *Any 2×2 matrix $\mathbf{M}$ with zero determinant (and non-zero trace T) has the property that*

$$\mathbf{M}^n = T^{n-1}\mathbf{M}, \qquad n \geq 1.$$

Proof. Since $\{w_n^*\}_0^\infty = \{w_n(0,1;T,0)\}_0^\infty = \{0,1,T,T^2,T^3,T^4,\ldots\}$ (that is to say, $w_n^* = T^{n-1}$ for $n \geq 1$), then Theorem 1.1 reads, for $n \geq 1$,

$$\begin{aligned} \mathbf{M}^n &= \begin{pmatrix} T^n - DT^{n-1} & BT^{n-1} \\ CT^{n-1} & T^n - AT^{n-1} \end{pmatrix} \\ &= T^{n-1}\begin{pmatrix} T-D & B \\ C & T-A \end{pmatrix} \\ &= T^{n-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= T^{n-1}\mathbf{M}, \end{aligned} \tag{C.1}$$

as required. □

Thus, for example,

$$\begin{aligned}\begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}^n &= 4^{n-1}\begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}, \\ \begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 3 \end{pmatrix}^n &= 5^{n-1}\begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 3 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Remark 2.2. It is immediate from Corollary 2.2 that any singular matrix $\mathbf{M}$ with a trace of unity has the property that $\mathbf{M}^n = \mathbf{M}$ for $n \geq 1$, and is idempotent as a consequence. Examples of such matrices are

$$\begin{pmatrix} 3-\sqrt{3} & \frac{3}{4} \\ 4(\frac{5}{\sqrt{3}}-3) & \sqrt{3}-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2(3+i) & 2(1+2i) \\ -7+3i & -(5+2i) \end{pmatrix}; \tag{2.10}$$

more generally, setting $C(A,B) = A(1-A)/B$ and $D(A) = 1-A$, then any matrix (1.1) of the form

$$\mathbf{M}(A,B) = \mathbf{M}(A,B,C(A,B),D(A)) = \begin{pmatrix} A & B \\ C(A,B) & D(A) \end{pmatrix} \tag{2.11}$$

has this property, of which those in (2.10) are merely two instances.

2.2 An Application: Polynomial Families

Let $A(x), B(x), C(x) \in \mathbb{Z}[x]$, and suppose the (ordinary) generating function $O(x)$ of a sequence of integers satisfies a general quadratic governing equation

$$0 = A(x)O^2(x) + B(x)O(x) + C(x). \tag{2.12}$$

The functional coefficients $A(x), B(x), C(x)$ can be considered to give rise to a family of associated polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, defined as

$$\begin{aligned} \alpha_n(x) &= \alpha_n(A(x),B(x),C(x)) \\ &= (1,0)\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0. \end{aligned} \tag{2.13}$$

The first few polynomials are easily computed from (2.13) as

$$\begin{aligned} \alpha_0(x) &= 1, \\ \alpha_1(x) &= -B(x), \\ \alpha_2(x) &= B^2(x) - A(x)C(x), \\ \alpha_3(x) &= 2A(x)B(x)C(x) - B^3(x), \\ \alpha_4(x) &= B^4(x) - 3A(x)B^2(x)C(x) + A^2(x)C^2(x), \\ \alpha_5(x) &= 4A(x)B^3(x)C(x) - 3A^2(x)B(x)C^2(x) - B^5(x), \end{aligned} \tag{2.14}$$

and so on. There are, of course, a great number of such polynomial families in existence due to the prevalence of integer sequences whose generating functions are quadratic in the way described. Previous work has detailed results on the specific instances of Catalan, (Large) Schröder and Motzkin polynomial families that are delivered by those familiar integer sequences forming a natural grouping in the context of lattice path counting (see, for example, works by the authors in [4-6] and references therein).

We are now in a position to offer a new interpretation of the generic polynomial family as defined, for directly from Theorem 1.1,

$$\begin{aligned} &\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \\ &\quad = \begin{pmatrix} w^*_{n+1}(x) & A(x)w^*_n(x) \\ -C(x)w^*_n(x) & w^*_{n+1}(x) + B(x)w^*_n(x) \end{pmatrix} \end{aligned} \tag{2.15}$$

for $n \geq 1$, where $\{w^*_n(x)\}_0^\infty$ can be termed a generalized Fibonacci sequence of polynomials

$$\{w^*_n(x)\}_0^\infty = \{w_n(0, 1; -B(x), A(x)C(x))\}_0^\infty. \tag{2.16}$$

Thus, (2.13) and (2.15) give immediately that (for $n \geq 0$)

$$\alpha_n(A(x), B(x), C(x)) = w^*_{n+1}(x) = w_{n+1}(0, 1; -B(x), A(x)C(x)), \tag{2.17}$$

and the polynomial $\alpha_n(A(x), B(x), C(x))$ is seen to be *but one member of a sequence of generalized Fibonacci polynomials characterized by* $A(x), B(x)$ *and* $C(x)$; the description is a new one for this class of polynomials.

2.3 Some Observations

We finish with some observations, and a remark. Firstly, in view of (1.2), and with $\alpha_0(x) = w^*_1(x) = 1$, further polynomials within any such family are delivered now by the recurrence

$$w^*_{n+2}(x) = -B(x)w^*_{n+1}(x) - A(x)C(x)w^*_n(x), \qquad n \geq 0, \tag{2.18}$$

as $\alpha_1(x) = w^*_2(x) = -B(x)w^*_1(x) - A(x)C(x)w^*_0(x) = -B(x)\cdot 1 - A(x)C(x)\cdot 0 = -B(x)$, and in turn $\alpha_2(x) = w^*_3(x) = -B(x)w^*_2(x) - A(x)C(x)w^*_1(x) = -B(x)[-B(x)] - A(x)C(x)\cdot 1 = B^2(x) - A(x)C(x)$, $\alpha_3(x) = w^*_4(x) = -B(x)w^*_3(x) - A(x)C(x)w^*_2(x) = -B(x)[B^2(x) - A(x)C(x)] - A(x)C(x)$ $[-B(x)] = 2A(x)B(x)C(x) - B^3(x)$, and so on, in agreement with (2.14), this recursive method for generating those polynomials of any given family not having been invoked before. Secondly, writing

$$\mathbf{P}(x) = \mathbf{P}(A(x), B(x), C(x)) = \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}, \tag{2.19}$$

then (2.18) is expressible immediately (by (2.13) and (2.17)) as the algebraic matrix identity

$$\begin{aligned} 0 &= (1,0)\mathbf{P}^{n+1}(x)(1,0)^T + \\ &\quad (B(x),0)\mathbf{P}^n(x)(1,0)^T + (A(x)C(x),0)\mathbf{P}^{n-1}(x)(1,0)^T \end{aligned} \tag{2.20}$$

(where T denotes transposition), which holds for $n \geq 0$. In the case $n = 0$, for example, it reads (with $\mathbf{I}_2$ the order 2 identity matrix)

$$\begin{aligned} 0 &= (1,0)\mathbf{P}(x)(1,0)^T + \\ &\quad (B(x),0)\mathbf{I}_2(1,0)^T + (A(x)C(x),0)\mathbf{P}^{-1}(x)(1,0)^T, \end{aligned} \tag{2.21}$$

and with

$$\begin{aligned} \mathbf{P}(x)(1,0)^T &= (-B(x), -C(x))^T, \\ \mathbf{I}_2(1,0)^T &= (1,0)^T, \\ \mathbf{P}^{-1}(x)(1,0)^T &= (0, 1/A(x))^T, \end{aligned} \tag{2.22}$$

the r.h.s. of (2.21) is $(1,0)(-B(x), -C(x))^T + (B(x),0)(1,0)^T + (A(x)C(x),0)$ $(0, 1/A(x))^T = -B(x) + B(x) + 0 = 0$. Noting that $\mathbf{P}^2(x)(1,0)^T = (B^2(x) - A(x)C(x), B(x)C(x))^T$, then for $n = 1$ the r.h.s is

$$\begin{aligned} &(1,0)\mathbf{P}^2(x)(1,0)^T + (B(x),0)\mathbf{P}(x)(1,0)^T + (A(x)C(x),0)\mathbf{I}_2(1,0)^T \\ &\quad = (1,0)(B^2(x) - A(x)C(x), B(x)C(x))^T \\ &\qquad\quad + (B(x),0)(-B(x), -C(x))^T + (A(x)C(x),0)(1,0)^T \\ &\quad = B^2(x) - A(x)C(x) - B^2(x) + A(x)C(x) \\ &\quad = 0; \end{aligned} \tag{2.23}$$

other cases for values of $n = 2, 3, 4, \ldots,$ are verified readily by computer.

Remark 2.3. The linear polynomial recursion (2.18) was in fact first seen in 2010 as Lemma 1.1 of [1, p. 10], and a simple proof given accordingly using matrices. The paper delivered some so called auto-identities for particular (that is, (Large) Schröder and Motzkin) polynomial families associated with their namesake sequences based on some theory developed for the general sequence-related polynomial $\alpha_n(x) = \alpha_n(A(x), B(x), C(x))$ (these were produced algebraically by computer from a suite of Householder root finding schemes used in numerical analysis), but at the time a connection between $\alpha_n(x)$ and the corresponding generalized Fibonacci polynomial $w^*_{n+1}(x) = w_{n+1}(0,1;-B(x),A(x)C(x))$ had not been made; it is emphasized that (2.18) emerges here completely independent of the derivation method, and the approach made, in [1].

In addition, Lemma 1.2 therein (p. 10 also) can be expressed here as the non-linear recurrence equation $0 = A(x)C(x)w^*_p(x)w^*_q(x) - w^*_{p+1}(x)w^*_{q+1}(x) + w^*_{p+q+1}(x)$, which holds for $p, q \geq 0$ (by way of example, setting $p = q = 1$, the r.h.s. is $A(x)C(x)w^{*2}_1(x) - w^{*2}_2(x) + w^*_3(x) = A(x)C(x)\cdot 1^2 - [-B(x)]^2 + B^2(x) - A(x)C(x) = 0$, while for $p = 3$, $q = 1$, we see that the r.h.s. is $A(x)C(x)w^*_3(x)w^*_1(x) - w^*_4(x)w^*_2(x) + w^*_5(x) = A(x)C(x)[B^2(x) - A(x)C(x)]\cdot 1 - [2A(x)B(x)C(x) - B^3(x)][-B(x)] + B^4(x) - 3A(x)B^2(x)C(x) + A^2(x)C^2(x) = \cdots = 0$).

Finally, another existing non-linear identity may be written in terms of our generalized Fibonacci polynomials. That of [4, p. 76] yields $A^n(x)C^n(x) = w^{*2}_{n+1}(x) + A(x)C(x)w^{*2}_n(x) + B(x)w^*_{n+1}(x)w^*_n(x)$ which, valid for $n \geq 1$, is easily checked by hand for a few low values of n.

To conclude matters, note that an alternative description of the general polynomial $\alpha_n(A(x),B(x),C(x))$ is available from Lemma 2.1, for writing $\mathbf{P}(x)$ (2.19) as

$$\mathbf{P}(x) = -C(x)\begin{pmatrix} B(x)/C(x) & -A(x)/C(x) \\ 1 & 0 \end{pmatrix}, \qquad (2.24)$$

then the lemma yields

$$\begin{aligned}\alpha_n(A(x),B(x),C(x)) & \\ = \; & [-C(x)]^n w_{n+1}(0,1;B(x)/C(x),A(x)/C(x)), \quad n \geq 0; \qquad (2.25)\end{aligned}$$

this, and (2.17), allow us to write down a previously unseen equivalence relation

$$\begin{aligned}[-C(x)]^n w_{n+1}(0,1;B(x)/C(x),A(x)/C(x)) & \\ = \; & w_{n+1}(0,1;-B(x),A(x)C(x)), \qquad (2.26)\end{aligned}$$

which holds for $n \geq 0$.

3 Summary

In this paper we have re-formulated previous results of J. Pla [9] and J. McLaughlin [8] for those entries of an arbitrary 2-square matrix when exponentiated, obtaining (with two proofs provided) closed forms in terms of a particular Horadam sequence. This allows for a new interpretation to be made for a class of polynomial families associated with integer sequences of a certain type, in so far as in any given family the polynomial members map to those of two (equivalent) corresponding generalized Fibonacci polynomial sequences.

We note, for completeness, that Pla [9] alluded to, but did not include, our inductive proof (Proof I) of Theorem 1.1. McLaughlin's own inductive argument in [8] itself relied on a constructive proof of the linear recurrence for $y_n(T, M)$ (which is merely (1.2) for $w_n^* = w_n(0, 1; T, M)$) by appealing to its definition (2.1) as a binomial coefficient sum.

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MSC2010: 11C20, 11K31