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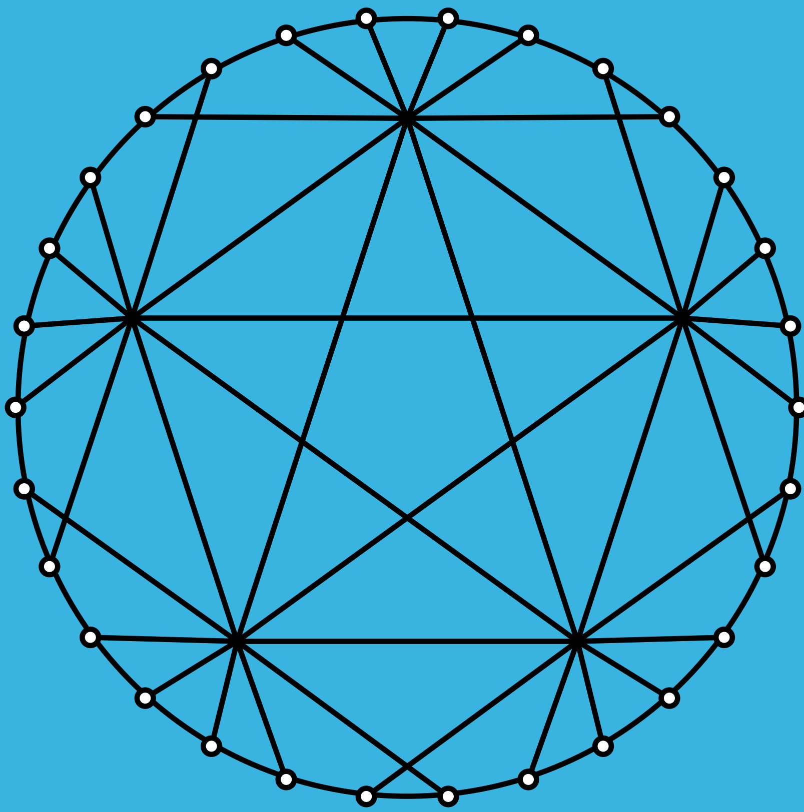

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Coloring graphs to produce walks without forbidden repeats

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Abstract. We consider the problem of coloring the edges of a graph such that every pair of vertices are joined by an ℓ-rainbow walk, that is, a walk where for every sub-walk of length at most $\ell+1$ its edges receive different colors. We show that the minimum number of colors needed is at most $2\ell+1$ if the graph has a cycle of length at least ℓ. We also provide a sharp bound for $\ell=2$ in bridgeless graphs, and general bounds for sufficiently large bridgeless graphs, and show that if the graph contains two sufficiently long edge-disjoint closed trails then the number of colors needed is at most $\ell+2$.

1 Introduction

There is a growing literature on coloring the edges of a graph so that there exist "connectedness substructures" without some forbidden pattern of colors. One version of the problem was introduced by Borozan et al. [2], who defined a coloring where every pair of vertices are joined by a path such that no two consecutive edges on the path have the same color. The associated parameter is called the path connection number. We considered the same coloring problem for walks [14] and for trails [9].

In this paper we consider a related coloring condition. Specifically, we define a walk to be *ℓ-rainbow* if for every sub-walk of length at most $\ell+1$, its edges receive different colors. That is, a color can only be re-used every $\ell+1^{\text{st}}$ edge. We will denote the condition by R_ℓ. Thus the original problem was about R_1. We define the *R_ℓ-walk connection number* as the minimum

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number of colors in a coloring of the edges of graph G such that every pair of vertices are joined by an R_ℓ-walk, and denote the parameter by $w_\ell(G)$.

This concept for paths was introduced by Li et al. [13] and Chartrand et al. [3, 6] (though the latter define it slightly differently). For example, Li et al. [13] showed that if a graph is 2-connected, then there is a coloring of the edges with at most 5 colors such that every pair of vertices are joined by an R_2-path. They also provided some calculations, including determining the value of the associated parameter for a complete bipartite graph, which is also given in [3]. The concept was renamed $(1, \ell)$-rainbow connectivity in the survey [11] and book [12], where they denote the associated parameter by $rc_{1,\ell}(G)$. Further, related questions about rainbow-colored paths and cycles were investigated by Chartrand et al. [4, 5]. Related work is considered in [13, 8, 15]. Note that there is also work on rainbow Hamilton cycles, but these are questions about all colorings rather than the existence of a coloring. And there is also work on graphs where the edges are already colored.

Our focus is on walks and in particular upper bounds on the parameter. The original paper contained the bounds for the proper-walk connection number (the 1-walk connection number in the terminology of this paper):

Theorem 1.1. [14]

(a) *If graph G is connected and not a tree, then $w_1(G) \leq 3$.*

(b) *If graph G is 2-edge-connected, then $w_1(G) \leq 2$.*
And these results are sharp.

An important paper on the walks case is the recent work by Bang-Jensen et al. [1], who gave an algorithmic characterization to determine which graphs have R_1-walk connection number 3.

We proceed as follows. In Section 2 we show that the R_ℓ-walk connection number is at most $2\ell + 1$ if the graph has a closed trail of length at least ℓ, and discuss the sharpness of this result and its extension. For bridgeless graphs, we show in Section 3 that the R_2-walk connection number is at most 4 and this is sharp, and in Section 4 that in general the R_ℓ-walk connection number is at most $(3\ell + 5)/2$ if the graph is sufficiently large. In Section 5 we consider graphs with two disjoint long closed trails and show that for such graphs, the R_ℓ-walk number at most $\ell + 2$. Finally in Section 6 we briefly consider some related problems.

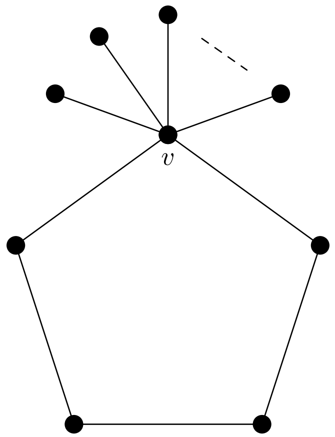

Figure 1: A graph with arbitrarily large w_6

2 Walks and closed trails

It is immediate that in a tree one cannot reverse a colored walk. Thus the R_ℓ-walk and R_ℓ-path connection numbers both equal the version of the edge-chromatic number where edges within distance ℓ must have different colors. For R_2, Li et al. [13] and Chartrand et al. [3] noted that this value equals one less than the maximum sum of degrees of adjacent vertices. The observation about trees carries over to graphs where every cycle is small. For example, define a pineapple by taking a cycle C_m and adding many leafs at one vertex, as shown in Figure 1. If $m < \ell$, then the walk from one end-vertex to another is short enough that both (and hence all) the leaf-edges must have different colors.

On the other hand, if there is a sufficiently long cycle, then the R_ℓ-walk connection number is bounded, as we now show. We will need bounds on the (vertex) chromatic number of the powers of cycles (a subset of the so-called Harary graphs). The exact values were given by Kazemi and Jalilolghadr [10]. Let C_n^ℓ denote the graph obtained from the cycle C_n by joining all pairs of vertices at distance at most ℓ.

Theorem 2.1. [10] *For $1 \le \ell \le n/2$, the chromatic number of C_n^ℓ is*

$$\chi(C_n^\ell) = \left\lceil \frac{n}{\lfloor n/(\ell+1) \rfloor} \right\rceil$$

From this one can deduce the following:

Lemma 2.2. *Let $2 \leq \ell \leq n/2$.*

(a) $\chi(C_n^\ell) \leq 2\ell + 1$.

(b) $\chi(C_n^\ell) = 2\ell + 1$ *exactly when* $n = 2\ell + 1$.

(c) $\chi(C_n^\ell) = 2\ell$ *exactly when* (i) $\ell = 2$ *and* n *is not a multiple of* 3*;* (ii) $\ell = 3$ *and* $n \in \{6, 11\}$*; and* (iii) $\ell \geq 4$ *and* $n = 2\ell$*.*

(d) *For fixed* ℓ*,* $\chi(C_n^\ell) \leq \ell + 2$ *for all* n *sufficiently large.*

Proof. One can readily prove these claims from scratch. But they also follow from Theorem 2.1. Say that $n = q(\ell + 1) + r$ where $0 \leq r \leq \ell$. Then we have that

$$\chi(C_n^\ell) = \ell + 1 + \left\lceil \frac{r}{q} \right\rceil.$$

Part (a) follows since $r \leq \ell$. Part (b) follows since the only way that $\lceil r/q \rceil$ can equal ℓ is that $r = \ell$ and $q = 1$. Part (d) is immediate as $q \geq \ell$ suffices.

For part (c) assume $\ell = 2$. Then $\lceil r/q \rceil = 1$ when $r \neq 0$ (except for the case covered by (b)). Assume $\ell = 3$. Then $r/q > 1$ when $q = 1$ and $r = 2$ and when $q = 2$ and $r = 3$. For $\ell \geq 4$, the only way that $r/q > \ell - 2$ is that $q = 1$. $\square$

The above result suggests considering cycles in the graph. But it is sufficient to consider closed trails: that is, closed walks all of whose edges are distinct.

Theorem 2.3. *If a connected graph* G *contains a closed trail of length* m *where* $m \geq \ell$*, then* $w_\ell(G) \leq \ell + \lfloor \chi(C_m^\ell)/2 \rfloor + 1$.

Proof. Let T be a closed trail of length m. Let F be a forest connecting T to the vertices not in T. We will color the edges of T and F such that between any two vertices u and v of G, the desired R_ℓ-walk can be obtained as follows. Let t_u be the vertex of T closest to u and t_v be the vertex of T closest to v, where possibly $t_u = u$, $t_v = v$, and/or $t_u = t_v$. The walk proceeds in F from u to t_u, traverses T completely back to t_u, then walks along T to t_v, and then proceeds in F to v.

First, we color the closed trail T such that every pair of edges up to distance ℓ apart (as measured in their appearance in T with wrap around) receive different colors. The number of colors needed equals the (ordinary vertex) chromatic number of the ℓ^{th} power of C_m. If $\ell > m/2$ then $\chi(C_m^\ell) = m <$

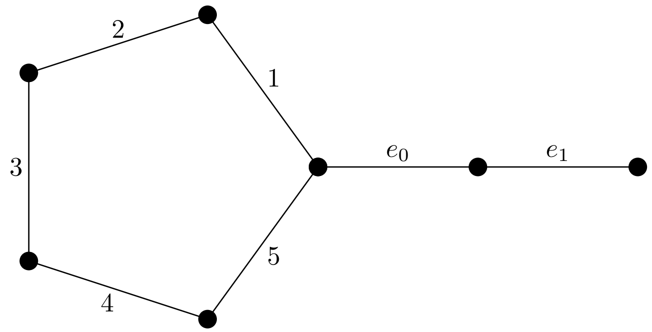

Figure 2: Coloring edges near the trail

2ℓ. Otherwise, by Lemma 2.2, $\chi(C_m^\ell) \leq 2\ell+1$. In particular this implies that $\lceil \chi(C_m^\ell)/2 \rceil \leq \ell+1$, and since $\lceil x/2 \rceil + \lfloor x/2 \rfloor = x$ for all integers x, rearranged this implies that $\chi(C_m^\ell) \leq \ell + \lfloor \chi(C_m^\ell)/2 \rfloor + 1$. In other words, we have enough colors to color the trail T.

We color the edges e of F in order based on their distance d from T (where an edge incident with T has distance 0). There are three cases. If $d \leq \ell - \lceil \chi(C_m^\ell)/2 \rceil$, then we give all edges at distance d the same color, in particular a new color not already used. See Figure 2 for an example for the case $\ell = 3$ and $d = 0$: edge e_0 receives the 6^{th} color. The number of new colors needed is at most $\ell - \lceil \chi(C_m^\ell)/2 \rceil + 1$. We have enough colors, since the total number of colors used is $\chi(C_m^\ell) + \ell - \lceil \chi(C_m^\ell)/2 \rceil + 1 = \ell + \lfloor \chi(C_m^\ell)/2 \rfloor + 1$.

If $\ell \geq d > \ell - \lceil \chi(C_m^\ell)/2 \rceil$, then for edge e, let the set $\mathcal{F}_e$ consist of the d edges joining e to T together with the $\ell - d$ edges in both directions on T. Note that $|\mathcal{F}_e| = 2\ell - d$. It suffices to color e so that it does not share a color with any edge in $\mathcal{F}_e$. See Figure 2 for an example for the case $\ell = 3$ and $d = 1$: edge e_1 can have color 3. Thus we can color edge e greedily provided $\ell + \lfloor \chi(C_m^\ell)/2 \rfloor + 1 > 2\ell - d$. Rearranged, it is sufficient that $d > \ell - \lceil \chi(C_m^\ell)/2 \rceil$.

Finally, if $d > \ell$, then we let $\mathcal{F}_e$ be the first ℓ edges on the path joining e to T, and give e any color not in $\mathcal{F}_e$. $\square$

As a consequence of Lemma 2.2(a) and Theorem 2.3, we obtain the main result:

Theorem 2.4. *If a connected graph G contains a closed trail of length at least ℓ, then $w_\ell(G) \leq 2\ell + 1$.*

The above theorem is best possible, as we now show. Define a *whiskered cycle* as the graph obtained from a cycle by attaching at least one end-vertex to every vertex of the cycle. We call the leaf-edges the *whiskers.*

Lemma 2.5. *Let G be a whiskered m-cycle with sufficiently many whiskers at each vertex.*

(a) *If m is not a multiple of 3, then $w_2(G) = 5$.*

(b) *If $m \in \{6, 7, 11\}$, then $w_3(G) = 7$.*

(c) *For $\ell \geq 4$, if $m \in \{2\ell, 2\ell + 1\}$, then $w_\ell(G) = 2\ell + 1$.*

Proof. Consider a valid coloring. The stated values are upper bounds by the above theorem; so it suffices to show that these many colors are required. Let C denote the cycle. If there are enough whiskers at some vertex v of C, then there must be two whiskers at v with the same color. To get between these two end-vertices, one must go around the cycle. Since this condition holds for each choice of vertex v, if follows that the cycle C must be colored such that any two edges at distance at most ℓ have different colors.

(a) We have $\ell = 2$. By Lemma 2.2 we know that there are (at least) four colors on C. Consider some vertex v of the cycle, and let e_1, e_2, e_3, e_4 denote the four consecutive edges in some orientation of C such that v is incident to e_2 and e_3. Suppose there is a repeated color in the collection e_1, e_2, e_3, e_4. Then it must be that e_1 and e_4 have the same color. But this pattern cannot hold all the way around the cycle C, since it would require every 3rd edge to have the same color, and m is not a multiple of 3. It follows that there must be a choice of vertex v where e_1, e_2, e_3, e_4 all get different colors. Hence the whiskers incident to v have four forbidden colors; and therefore need a fifth color.

(b,c) We have $\ell \geq 3$. If $m = 2\ell + 1$ then we are done, as one needs $2\ell + 1$ colors for the cycle. If $m = 2\ell$, then it follows that the every whisker needs a color not used on C. Since all the edges of C receive different colors, this means that one needs $2\ell + 1$ colors. Finally, consider the case that $\ell = 3$ and $m = 11$. In the coloring of the edges of the cycle C, it can be checked by hand or by computer that for all valid 6-colorings of C there must exist six consecutive edges that receive different colors. Thus one needs a seventh color for the whiskers incident with the middle vertex of that sextet. □

One can obtain a slightly stronger conclusion for graphs without end-vertices. We define an *Eulerian tour* of graph G as a closed walk that uses every edge of G at least once. If the edges are colored, then an R_ℓ*-tour* means an Eulerian tour such that every at most $\ell+1$ consecutive edges have different colors, including wrap around. (That is, the walk can be continued around the tour again and remain ℓ-rainbow.) Having an R_ℓ-tour is a stronger condition than having the desired walks.

Theorem 2.6. *Let G be a connected graph with minimum degree at least 2 that contains a closed trail of length at least ℓ. Then there is an $(2\ell+1)$-edge-coloring of G such that there is an R_ℓ-tour of G.*

Proof. Let T be the closed trail of length at least ℓ. Define G' as in the proof of Theorem 2.3 to be a spanning subgraph with the edges of T and a forest connecting every other vertex to the trail. Color G' as in that proof. Choose an orientation of T.

Consider some edge e in G but not G'. There are three cases: (i) Assume $e = xy$ where both x and y are on T. Then let $\mathcal{F}_e$ consist of the ℓ consecutive edges of T immediately after (some occurrence of) y together with the ℓ consecutive edges of T immediately before (some occurrence of) x. (These may overlap.) Give e any color not used on $\mathcal{F}_e$.

(ii) Assume $e = xv$ where x is on T but v is not. Then let $\mathcal{F}_e$ consist of the ℓ consecutive edges starting in the forest at v and going onto T, together with the ℓ consecutive edges of T immediately before (some occurrence of) x. Give e any color not used on $\mathcal{F}_e$.

(iii) Assume $e = uv$ where neither u nor v is on T. Then let $\mathcal{F}_e$ consist of the ℓ consecutive edges starting in the forest at v and going onto T, together with the ℓ consecutive edges starting in the forest at u and going to T but reverse along T. Give e any color not used on $\mathcal{F}_e$.

We claim this coloring yields the desired R_ℓ-tour. Our strategy is to start on the closed trail and walk around it and pick up every edge in turn. To pick up an edge of type (i), one can just traverse it and keep walking around T. To pick up an edge of type (ii), exit T at (the chosen occurrence of) x, traverse the edge, and then walk back down to T and continue circulating. To pick up an edge of type (iii), exit T along the forest-path up to u, then traverse the edge uv and then walk back down to T.

Finally, we note that the above process ensures that each forest edge f (that is, an edge of G' not in T) is traversed. To see this, let u be an end-vertex of G' such that the path in G' from u to T contains edge f. Then, since the graph G has minimum degree at least 2, the vertex u has an edge of type (ii) or (iii), and f is traversed when that edge is picked up. $\square$

Of course the condition on the minimum degree is necessary for an R_ℓ-tour. Further, cycles of length at most ℓ do not suffice. To see this, consider for example the barbell consisting of two disjoint such cycles and a path joining them.

3 The R_2-walk connection number in bridgeless graphs

Theorem 2.4 shows that the R_2-walk connection number of a cyclic connected graph is at most 5. We show here that if the graph is bridgeless then the maximum is actually 4.

We need a new idea. Consider a subgraph Q. We say that an Eulerian tour W_Q of Q is an *ℓ-reversible tour* if for every vertex v in Q there exists a sub-trail S_v of length ℓ starting at v such that W_Q also includes the reverse of S_v. Note that an R_ℓ-coloring of W_Q requires at least $\ell+1$ colors, such W_Q must have at least this many edges.

The first result provides the fundamental connection between reversible tours and walk colorings that "improves" on Theorem 2.3

Theorem 3.1. *Let G be a connected graph. Assume G has a subgraph Q with an ℓ-reversible tour W_Q with an R_ℓ-coloring using k colors. Then $w_\ell(G) \le k$.*

Proof. Consider the coloring of Q. We noted above that $k \ge \ell+1$. Let F be a forest such that every vertex not on Q is joined to Q by a path in F. For each vertex v of Q, let S_v be the sub-trail of W_Q of length ℓ starting at v. We color the edges of F in order of their distance from Q. For each such edge e, let v_e be the closest vertex of Q. Then consider a walk from e to Q and then along S_{v_e}. Define $\mathcal{F}_e$ as the first ℓ edges of this walk. Give e any color that does not appear in $\mathcal{F}_e$.

We claim that this coloring provides the desired R_ℓ-walks. Suppose that one wants to walk from vertex x to vertex y. Let v_x and v_y denote the closest vertices of Q. Start at x and walk to Q (if necessary) using the edges of

F; then proceed along S_{v_x} and continue along W_Q until it traverses S_{v_y} in reverse; and then (if necessary) walk to y using the edges of F. $\square$

One natural idea for a suitable subgraph Q is a *barbell*: two edge-disjoint cycles joined by a (possibly trivial) path. Indeed, two edge-disjoint closed trails are just as good. This approach for R_1 was used in the original paper, where it was noted that one needs cycles of suitable parity:

Lemma 3.2. [14] *If a connected graph G has two edge-disjoint odd cycles, then G has a barbell Q that can be 2-colored to provide a 1-reversible tour.*

We consider now the idea for R_2. We will need the following well-known fact (given for example as Fact 6 of section 4.8.1 of [16]):

Lemma 3.3. *If x and y are positive integers with gcd 1, then every positive integer at least $(x-1)(y-1)$ can be expressed as a nonnegative integer combination of x and y.*

Lemma 3.4. *Let G be a connected graph that contains two edge-disjoint closed trails each of length at least 4. Then G contains a subgraph Q that has an 2-reversible tour with an R_2-coloring using 4 colors.*

Proof. Let Q be the subgraph obtained by taking the two edge-disjoint closed trails, say C_1 and C_2, and some path P from v_1 to v_2 that joins them (possibly $v_1 = v_2$).

We will color each closed trail C_i such that the two edges incident with v_i receive the same color and there is a R_2-walk from v_i to v_i around C_i. Say C_i has m edges. Then one way to proceed is to use some combination of the patterns a, b, c and a, b, c, d to color the first $m-1$ edges and then use a for the last edge. By Lemma 3.3 this approach works except when $m-1 = 5$. But then the coloring a, b, c, d, b, a works.

First color P arbitrarily so that one has an R_2-walk from one end to the other. We need to synchronize the colorings of the C_i with P. For each i, define g_i as the color that the two edges incident with v_i will receive. We first determine the colors g_1 and g_2. If P has at least two edges, then choose g_1 to be different from the colors of the first two edges of P and g_2 to be different from the colors of the last two edges of P. If P has only one edge, then choose g_1 and g_2 to be different from each other and from the color on P. And if P has no edge, then simply choose g_1 and g_2 to be different.

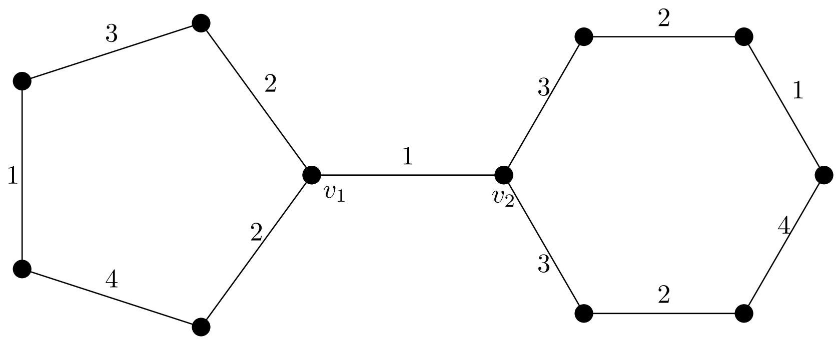

Figure 3: A coloring with a 2-reversible tour

Second, let p_i be the color of the edge of P incident with v_i if P has an edge; and let p_i be g_{3-i} otherwise. Then for each i, choose the color(s) of the second and penultimate edge of C_i to be different from p_i. See Figure 3 for an example of such a coloring.

This gives one the desired coloring. For a 2-reversible tour of Q with R_2-coloring, start by walking along P, go around C_2 in one direction, go back along P, go around C_1 in one direction, then again along P but then traverse C_1 and C_2 in the other direction. $\square$

What do graphs look like that don't have two edge-disjoint cycles? For example, Erdős and Pósa [7] noted that the number of edges is at most 3 more than the order. This suggests the following fact is probably known:

Lemma 3.5. *Let G be a 2-connected graph that does not contain two edge-disjoint cycles. Then G has a spanning subgraph that is either a Hamilton path or a subdivision of K_4 or $K_{3,3}$.*

Proof. Consider a cycle H_1. If this contains all vertices of G, we are done. So assume there is a vertex not on H_1. This vertex has two internally-disjoint paths to H_1, and so one obtains a subgraph H_2 of G that is a subdivision of $K_4 - e$.

If H_2 contains all vertices of G, then it contains a Hamilton path and we are done. So assume there is a vertex not on H_2. Then the vertex has two internally disjoint paths to the subgraph H_2. If these paths use either of the degree-3 vertices of H_2, then it is readily checked that there are two edge-disjoint cycles. So these paths do not use these vertices, and one has a subgraph H_3 of G that is a subdivision of K_4.

If H_3 is all of G then we are done. So assume there is some vertex outside H_3. Then this vertex has two internally disjoint paths to H_3. This yields two edge-disjoint cycles except in the case that these paths meet H_3 at degree-2 vertices that are on paths that don't share an endpoint. This yields a subdivision of $K_{3,3}$.

It can then be checked that the existence of a vertex outside the $K_{3,3}$ creates a second cycle. $\square$

As a consequence we obtain the following result:

Theorem 3.6. *If a graph G is 2-edge-connected, then $w_2(G) \leq 4$.*

Proof. If the graph has a Hamilton path, then three colors are needed to color the path to provide the desired walks. If the graph has a triangle, then the upper bound follows from Theorem 2.3. If the graph has two edge-disjoint cycles, then the upper bound follows from Theorem 3.1 and Lemma 3.4. So assume the graph has none of this. In particular, this means that there is no cut-vertex. By the above lemma, the graph is a subdivision of K_4 or $K_{3,3}$.

Consider first the case that G is a subdivision of K_4. Let C be a cycle that contains all four vertices of degree 3, say v_1, v_2, v_3, v_4. Suppose that C is a 5-cycle, say with v_1, v_2, v_3, v_4 as subpath. Then by the lack of triangles, both the path from v_1 to v_3 outside C and the path from v_2 to v_4 outside C must contain at least one internal vertex. Thus it is possible to choose C to not be a 5-cycle.

Then let P_1 be the path from v_3 to v_1 without the first vertex, and let P_2 be the path from v_4 to v_2 without the first vertex (so that a spanning subgraph consisting of C with P_1 and P_2 dangling off it). Color the cycle C with four colors so that one can circulate around it. (This is possible by Lemma 2.2.) For each path P_i, color it such that one can walk along it and join C and go around C the long way round to the other path (that is, using v_3 and v_4). This allows one to go down P_1 and up P_2. Thus this is a valid R_2-coloring.

Consider second the case that G is a subdivision of $K_{3,3}$. Let C be a cycle that contains all six vertices of degree 3, say meeting them in the order

$v_1, \dots, v_6$. Then for $i = 1, 2, 3$ let P_i be the path from v_{3+i} to v_i without the first vertex. If the set $\{v_1, \dots, v_6\}$ is not independent, then one can choose C such that P_3 has no edge. Then one can proceed as in the case that G is a subdivision of K_4. So assume that none of the v_i are adjacent. It follows that C has length at least 12. Color the cycle C with four colors so that one can circulate around it. (Possible by Lemma 2.2.)

Let e_1, e_2, e_3, e_4 be the two edges before and after v_2 on C (that is, v_2 is incident to e_2 and e_3). We claim that one can choose the coloring of C such that e_1 and e_4 have the same color; for if every choice of e_1 and e_4 have different colors, then the coloring of C must look like $1, 2, 3, 4, 1, 2, 3, 4, \dots$ Since C is long enough we can choose a coloring that does not have the property that every fifth edge has the same color. (For example, for the case of length 12 one can use four copies of the pattern $1, 2, 3$.)

Color P_2 as follows. Give the edge of P_2 incident to v_2 the color not present on any of the edges e_i, give the next edge of P_2 the color of e_1 and e_4, and then proceed along P_2. This ensures that one can walk down P_2 and join C and go either way around C. For paths P_1 and P_3, color them so that one can walk along down P_1 go around C in some direction and then go up P_3. This also provides the walk from P_1 or P_3 to P_2. □

3.1 Rosary graphs

The value 4 in Theorem 3.6 is best possible, as we now show. Define a *rosary graph* as follows. Take three cycles C_1, C_2, C_3 each of length $2a$. For each cycle, choose one pair of antipodal vertices d_i, d'_i ($i = 1, 2, 3$). Then join d'_i to d_{i+1} (arithmetic modulo 3) with a path P_i that has at least one internal vertex. Let b denote the total number of vertices added. An example with $a = 5$ and $b = 4$ is shown in Figure 4.

Lemma 3.7. *If $a \geq 5$, and neither a nor b is a multiple of 3, then the rosary graph has $w_2 \geq 4$.*

Proof. Suppose there is a valid 3-coloring of the edges. The condition on a and b means that none of the cycles in the graph has length a multiple of 3. Note that every R_2-walk must be periodic, that is, colored $1, 2, 3, 1, 2, 3, \dots$

Since $a \geq 5$, there exists a pair of antipodes c_1, c'_1 of C_1 that are not adjacent to d_1, d'_1. Let x be the penultimate vertex on P_3, and y the second vertex on P_1. We claim that for some choice of $\alpha \in \{c_1, c'_1\}$ and $\beta \in \{x, y\}$, there

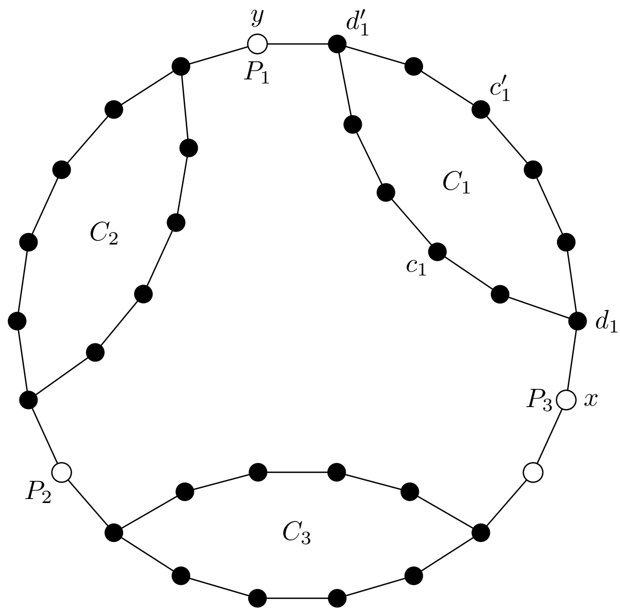

Figure 4: A block with $w_2 = 4$

is an R_2-walk between α and β going the long way round, meaning using each of P_1 through P_3. This is immediate if the R_2-walk between c_1 and c_1' goes the long way round, as it then must use both x and y.

So assume without loss of generality that the R_2-walk from c_1 to c_1' stays within the first cycle C_1 and uses d_1. Consider the two edges before and after d_1 on the walk. Since the edge d_1x shares a color with one of these edges, in the graph the R_2-walk from one of c_1, c_1', say c_1, cannot get onto P_3 there. Assume the R_2-walk from c_1 to x uses only the first cycle. Then the walk c_1 to d_1' to c_1' to d_1 to c_1 is R_2-colored and hence must be periodic. But since a is not a multiple of 3 and $a \geq 5$, there is a choice of antipodes and so one can choose vertex c_1 that does not have this property. It follows that we may assume the R_2-walk from c_1 to x goes the long way around.

That is, there is a R_2-walk from C_1 to C_1 using all the P_i and half of each of the other cycles. The same argument holds for each cycle C_2 and C_3 in turn. These colorings overlap and so give a proper R_2-coloring of a long cycle. But this presents a contradiction, since the long cycle has length $b + 3a$, which is not a multiple of 3. $\square$

4 General bridgeless graphs

The approach of Section 2 can provide an upper bound for graphs with long closed trails.

We show next that 2-edge-connected graphs whose longest closed trail is bounded have bounded order. It is probable that this fact has been noted before, and maybe with a realistic upper bound.

Lemma 4.1. *For all t, there are finitely many 2-edge-connected graphs whose maximum closed trail length is at most t.*

Proof. The proof is by induction on t. The base case is immediate; for example for $t = 3$ the only graph is K_3. So consider a longest closed trail T in graph G.

We claim that the longest closed trail in $G - E[T]$ has length less than t. Consider a longest closed trail T' that is edge-disjoint from T; say of length t'. By the maximality of T, the trails T and T' have no vertex in common. By the connectivity of G, there are two edge-disjoint paths from T' to T. These two paths can be combined with at least half of each of T and T' to get a closed trail in G of length at least $t/2 + t'/2 + 2$. It follows that $t' \le t - 4$, which proves the claim.

Now, for each vertex v outside T, there exist two edge-disjoint paths to T; choose one such pair and call this the *v-splay.* The v-splay must end in different vertices of T, else we have a contradiction of the maximality of T. Further, these ends cannot be consecutive on T, else we again get a contradiction. For each pair P of vertices of $V(T)$ that are not joined by an edge of T, define the auxiliary subgraph H_P that consists of all v-splays with that pair of terminals, and adding the edge e_P joining P (if necessary).

Assume the graph H_P is not K_2. Then the removal of a single edge cannot disconnect H_P, because the two vertices of P are still joined (either by their edge or by a splay), and every other vertex can get to at least one vertex in P. That is, H_P is 2-edge-connected. Consider a closed trail Q in H_P. If Q does not use e_P, then Q is in $G - E[T]$ and so has length less than t by the above claim. If Q does use e_P, then let Q' be the closed trail in G that is formed by replacing the edge e_P by a segment of T. It follows that the length of Q' is more than the length of Q, while Q' has at most t edges. Hence the longest closed trail in H_P has length at most $t - 1$.

By the induction hypothesis, it thus follows that the graph H_P has bounded order. Every vertex of G is in at least one of the auxiliary graphs H_P. Since there are $(t-1)(t-2)/2$ choices for P, it follows that G has bounded order. □

And so we get the following result:

Theorem 4.2. *If a graph G is 2-edge-connected and sufficiently large, then $w_\ell(G) \leq \ell + 2 + \lceil \ell/2 \rceil$.*

Proof. By Lemma 2.2, there is a t_0 such that $\chi(C_t^\ell) \leq \ell + 2$ for all $t \geq t_0$. By Lemma 4.1, if G is sufficiently large it contains a closed trail of length at least t_0. By Theorem 2.3 the bound follows. □

We do not know if this is even close to best possible, neither the value, nor whether the requirement that G be sufficiently large is necessary. It does seems likely that there is a suitable version R_ℓ of a rosary graph where $w_\ell(R_\ell) \geq (1+\epsilon)\ell$, but we are unable to prove this.

5 Two disjoint closed trails

We saw above that having a long closed trail improves the upper bound on the R_ℓ-walk number from $2\ell + 1$ to $3\ell/2 + 5/2$. We show next that having two long edge-disjoint closed trails improves the upper bound to $\ell + 2$. We will need the following lemma.

Lemma 5.1. *Fix some ℓ. If n is sufficiently large, then the path on n edges has an R_ℓ-walk coloring using $\ell + 2$ colors such that the first $\ell - 1$ edges and the last $\ell - 1$ edges have the same colors but reversed.*

Proof. Let F_0 denote the pattern of colors $1, 2, \ldots, \ell - 1$. Let X denote the other three colors (that is, $\ell, \ell+1, \ell+2$). We start with a color scheme of the form $F_0, X, F_1, X, F_2, X \ldots$, where each F_i is a permutation of the colors from 1 up to $\ell - 1$. The key is that, for each pair of consecutive F_i's, the colors are rearranged slightly, with each color moving at most one position. So, for example, by doing what computer science calls a bubble sort, after sufficiently many steps, the ordering is reversed. For instance, if $\ell = 4$ the coloring might start

```
1 2 3 x x x 1 3 2 x x x 3 1 2 x x x 3 2 1
```

Once the F_0 has been reversed to F_0^R, then if sufficiently long one can by Lemma 3.3 write the remaining number of edges as an integer combination of $\ell+1$ and $\ell+2$. Then the remaining edges of the path are colored with some combination of the patterns XF_0^R or $X'F_0^R$ where X' is $\ell, \ell+1$ say. $\square$

Theorem 5.2. *If connected graph G has two sufficiently long edge-disjoint closed trails, then it has a subgraph Q that has an ℓ-reversible tour with an R_ℓ-coloring using $\ell+2$ colors.*

Proof. Say the two edge-disjoint closed trails are C_1 and C_2. Let P be a path with ends v_1 to v_2 that joins C_1 and C_2 (possibly $v_1 = v_2$). Let Q be the subgraph with the edges of P, C_1, and C_2 and the incident vertices. The goal is to color Q such that it is possible to start on P and walk along, go around the closed trail C_1 in either direction, go back along P, around the closed trail C_2 in either direction, and so on. By the above lemma, one can give the cycles a suitable coloring. Further, one can choose the coloring of the cycles and the path such that the $\ell-1$ edges of C_1 ending at v_1, followed by P, followed by the $\ell-1$ edges of C_2 starting at v_2, form an ℓ-rainbow walk. $\square$

It seems possible that every sufficiently large 3-connected graph has two arbitrarily long disjoint closed trails. This would imply that every sufficiently large 3-connected graph has R_ℓ-walk number at most $\ell+2$.

6 Other comments

We have so far looked at the question of strengthening the R_1 condition. In another direction, one could for example consider the problem where the color cannot be the same for three consecutive steps. But here two colors is enough for path connection: simply take a spanning tree, designate a root, and color the edges of the tree based on the parity of their distance from the root.

For future work, one could consider stronger restrictions, such as that every two vertices are in a cycle, or there are multiple "disjoint" walks/trails/paths between them, as for example investigated in [8, 13]. One can also consider an upper bound on the length of the walk, as was done for paths in [15].

The complexity question for w_ℓ for $\ell \geq 2$ is open.

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