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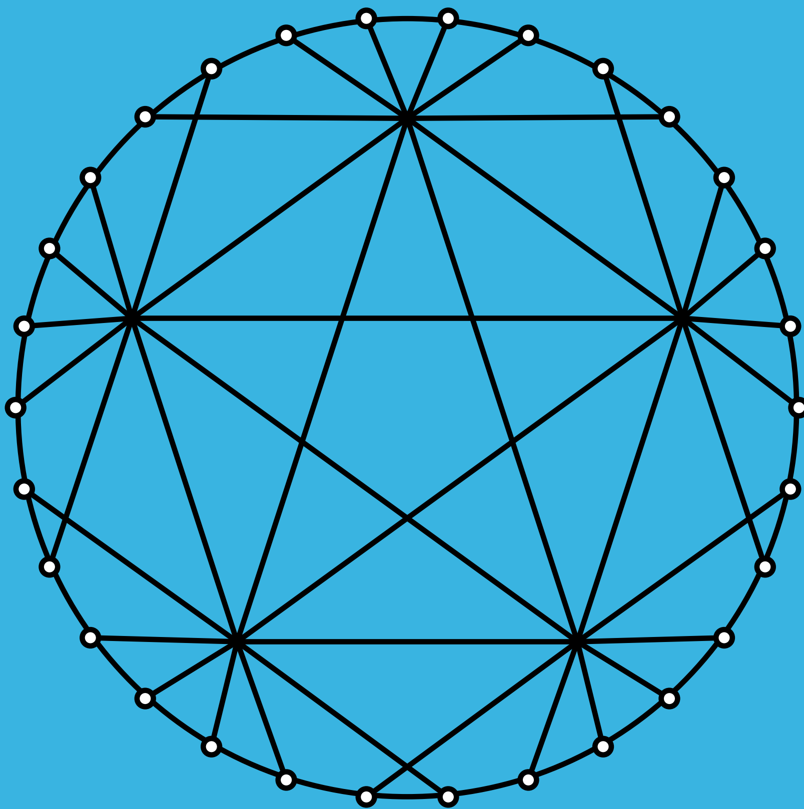

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Kings in products of digraphs

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Abstract. A k-king in a digraph D is a vertex that can reach every other vertex in D by a dipath of length at most k. Here, we investigate the existence of k-kings in products of digraphs. We show that for the Cartesian, strong and lexicographic products, a king exists in the product if and only if the kings exist in the factors. For the direct product, we show that the kings exist in the factors if they exist in the product.

1 Introduction

A *k-king* in a digraph D is a vertex that can reach every other vertex in D by a directed path (*dipath*) of length at most k. A *king* is a vertex that is a k-king for some k. Kings have been studied extensively in the past, especially in *tournaments*, i.e., digraphs in which every pair of distinct vertices is joined by exactly one arc [12, 17, 15, 8, 9, 10, 14, 11]. Also, [16] extended some of the results from tournaments to oriented graphs, i.e., digraphs with no symmetric pairs of directed arcs and without loops.

In this paper, we are primarily concerned with the kings in products of digraphs [5]. There are many types of product graphs, but only four products (direct, Cartesian, strong, and lexicographic) are associative and have the property that the canonical projections from the product to the factors are homomorphisms [6]. Each product has a long history with numerous applications [1, 2, 18, 4, 7]. Our main result characterizes the relationship between the kings in the product and the kings in the factors.

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Theorem 1.1. *Let D and D' be digraphs. Then:*

1. *(v, v') is a l-king in the Cartesian product $D \Box D'$ if and only if v is k-king in D and v' is a k'-king in D', where $k + k' = l$.*
2. *(v, v') is a k-king in the strong product $D \boxtimes D'$ if and only if v a k-king in D and v' is a k-king in D'.*
3. *(v, v') is a k-king in the lexicographic product $D \circ D'$ if and only if v is k-king in D, and either (a) v' is a k-king in D' or (b) v lies on a directed cycle of length at most k.*
4. *If (v, v') is a k-king in the direct product $D \times D'$, then v is a k-king in D and v' is a k-king in D'.*

This theorem appears in [5], without proof, citing this paper as a preprint.

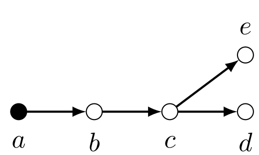
We note that the reverse implication for the direct product has been established in [13]. The proof is long and quite elaborate.

Theorem 1.2 ([13])**.** *(v, v') is a king in $D \times D'$ if and only if v is a king in D, v' is a king in D', and $\gcd(g_D(v), g_{D'}(v')) = 1$ where $g_D(v)$ is the greatest common divisor of the lengths of all closed directed walks in D containing the vertex v.*

In the rest of the paper we first define the necessary notions and terminology and then we prove Theorem 1.1.

2 Preliminaries

A *digraph*, or *directed graph*, is an ordered pair $D = (V, A)$ where V is a finite nonempty set of *vertices* and A is a set of ordered pairs of distinct vertices in V called *arcs*. A (v, w)-*diwalk* is a sequence of vertices $W = vv_1v_2 \dots v_{k-1}w$ such that vv_1, $v_{k-1}w$, and v_iv_{i+1} are arcs in D for each $1 \le i \le k-1$. The *length* of the above (v, w)-diwalk W is k. A (v, w)-*dipath* is a (v, w)-diwalk in which no vertices or edges are repeated. A *dicycle* is a closed dipath, that is, a (v, v)-dipath for some vertex v in D. A *king* is a vertex that can reach every other vertex in D by a dipath. A k-*king* is a vertex that can reach every other vertex in D by a dipath of length at most k. In the example below, vertex a is a 3-king.



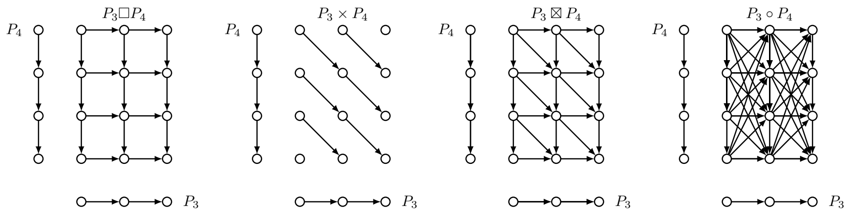
As discussed in [5], the four standard digraph products of D and D' are the *Cartesian product* $D \square D'$, the *direct product* $D \times D'$, the *strong product* $D \boxtimes D'$, and the lexicographic product $D \circ D'$. The vertex set of each product is given by

$$V = V(D) \times V(D') = \{(v, v'); v \in V(D) \text{ and } v' \in V(D')\}.$$

The arcs are given by

$$\begin{aligned} A(D \square D') &= \{(x, x')(y, y') | xy \in A(D), x' = y', \text{ or } x = y, x'y' \in A(D')\}, \\ A(D \times D') &= \{(x, x')(y, y') | xy \in A(D) \text{ and } x'y' \in A(D')\}, \\ A(D \boxtimes D') &= A(D \square D') \cup A(D \times D'), \\ A(D \circ D') &= \{(x, x')(y, y') | xy \in A(D), \text{ or } x = y \text{ and } x'y' \in A(D')\}. \end{aligned}$$

In each case, D and D' are called *factors* of the product. The figure below illustrates the products in the case of paths P_3 and P_4.



Cartesian product Direct product Strong product Lexicographic product

3 Proof of the main result

The proof of Theorem 1.1, specifically the forward implications that the existence of a king in the product implies the existence of the kings in the factors, relies on the following Lemma.

Lemma 3.1 (Proposition 1.3.2, [3]). *Let D be a digraph and let v, w be a pair of distinct vertices in D. If D has a (v, w)-diwalk W, then D contains a (v, w)-dipath P such that $A(P) \subseteq A(W)$. If D has a closed (v, v)-diwalk W, then D contains a dicycle C through v such that $A(C) \subseteq A(W)$.*

3.1 Kings in Cartesian products

The proof of the first equivalence in Theorem 1.1 will be established by the following two Lemmas.

Lemma 3.2. *If v is a k-king in a digraph D and v' is a k'-king in a digraph D', then (v, v') is a $(k + k')$-king in the Cartesian product $D \Box D'$.*

Proof. Pick any $w \neq v$ in D and any $w' \neq v'$ in D'. Since v is a k-king in D, there is an (v, w)-dipath $vv_1v_2 \dots v_{n-1}w$ of length $n \leq k$ in D. Similarly, since v' is k'-king in D', there is an (v', w')-dipath $v'v'_1v'_2 \dots v'_{n'-1}w'$ of length $n' \leq k'$ in D'.

Thus, $(v, v')(v_1, v')(v_2, v') \dots (v_{n-1}, v')(w, v')$ is a dipath in $D \Box D'$ connecting (v, v') and (w, v'). Similarly, $(w, v')(w, v'_1)(w, v'_2) \dots (w, v'_{n'-1})(w, w')$ is a dipath in $D \Box D'$ connecting (w, v') and (w, w').

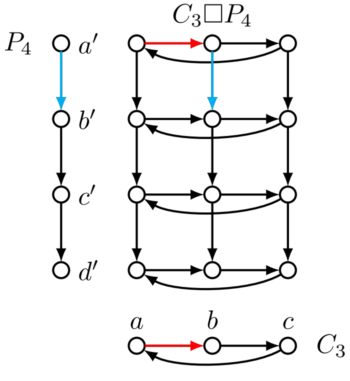
The two dipaths in $D \Box D'$ intersect only at (w, v'). Thus, the concatenated sequence

$$(v, v')(v_1, v')(v_2, v') \dots (v_{n-1}, v')(w, v')(w, v'_1)(w, v'_2) \dots (w, v'_{n'-1})(w, w')$$

is a dipath of length $n + n' \leq k + k'$ connecting (v, v') and (w, w').

Now, if either $w = v$ or $w' = v'$, the proof is analogous with an exception of a small detail that either the (v, w)-dipath or the (v', w')-dipath is of length 0. $\square$

The procedure from the proof of Lemma 3.2 is illustrated below for $D = C_3$ and $D' = P_4$.



The red dipath ab in C_3 lifts to a red dipath $(a,a')(b,a')$ in $C_3\Box P_4$. Similarly, the cyan dipath $a'b'$ in P_4 lifts to a cyan dipath $(b,a')(b,b')$ in $C_3\Box P_4$. Concatenating the two paths in $C_3\Box P_4$ together yields the dipath $(a,a')(b,a')(b,b')$ connecting (a,a') (b,b').

Lemma 3.3. *If (v,v') is an l-king in the Cartesian product $D\Box D'$ of digraphs D and D', then v is a k-king in D and v' is a k'-king in a digraph D' with $k+k'\leq l$.*

Proof. Let $w\neq v$ be a vertex in D and $w'\neq v'$ be a vertex in D'. Since (v,v') is an l-king, there exists a dipath

$$P=(v,v')(v_1,v_1')(v_2,v_2')(v_3,v_3')\ldots(v_{n-1},v_{n-1}')(w,w')$$

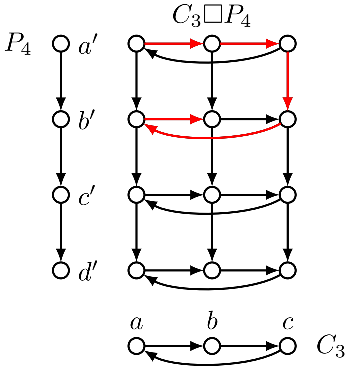
in $D\Box D'$ of length $n\leq l$ connecting (v,v') and (w,w').

Consider the sequence $vv_1v_2\ldots v_{n-1}w$. From the definition of the Cartesian product, any two consecutive vertices in the sequence are either identical or connected by an arc in D. Hence, if we remove all possible immediate duplications, we will have an (v,w)-diwalk in D of length $m\leq n$. By Lemma 3.1, there is an (v,w)-dipath of length at most $k\leq m\leq n$ in D. Thus, v is a k-king for $k\leq m\leq n$.

Similarly, v' is a k'-king in D' for $k'\leq m'\leq n$.

Note that $n-m$ is the number of times the dipath P in $D\Box D'$ moved vertically, i.e., the cardinality of the set $\{i|0\leq i\leq n-1; v_i=v_{i+1}\}$. Similarly, $n-m'$ is the number of time the path in $D\Box D'$ moved horizontally, i.e., the cardinality of the set $\{i|0\leq i\leq n-1; v_i'=v_{i+1}'\}$. Thus, $(n-m)+(n-m')=n$ and thus $m+m'=n$. Consequently, $k+k'\leq m+m'=n\leq l$. $\square$

The procedure from the proof of Lemma 3.3 is illustrated below.



The dipath $P = (a,a')(b,a')(c,a')(c,b')(a,b')(b,b')$ (in red) connects the 5-king (a,a') with a vertex (b,b') in $C_3 \square P_4$. The projection of this dipath into C_3 results in a sequence $abccab$ of vertices in C_3. Removing one of the duplicated c's yields an (a,b)-diwalk $abcab$ of length $m = 4$. Lemma 3.1 then guarantees the existence of an (a,b)-dipath ab of length $k = 1$. Similarly, projecting P into P_4 results in a sequence $a'a'b'b'b'$. Removing three vertices (one duplicated a' and two duplicated b''s) yields an (a',b')-diwalk $a'b'$ which already is a dipath of length $k' = m' = 1$.

3.2 Kings in strong products

The second equivalence of Theorem 1.1 will be established by the following two Lemmas.

Lemma 3.4. *If v is a k-king in a digraph D and v' is a k'-king in a digraph D', then (v,v') is a $\max\{k,k'\}$-king in the strong product $D \boxtimes D'$.*

Proof. Pick any vertex $w \neq v$ in D and any $w' \neq v'$ in D'. Since v is a k-king in D, there is a (v,w)-dipath $vv_1v_2 \dots v_{n-1}w$ of length $n \leq k$ in D. Similarly, since v' is k'-king in D', there is a (v',w')-dipath $v'v'_1v'_2 \dots v'_{n'-1}w'$ of length $n' \leq k'$ in D'. Without loss of generality, assume that $n \leq n'$. Thus, $(v,v')(v_1,v'_1)(v_2,v'_2)\dots(v_{n-1},v'_{n-1})(w,v'_n)$ is a dipath in $D \boxtimes D'$ connecting (v,v') and (w,v'_n). Furthermore,

$$(w,v'_n)(w,v'_{n+1})(w,v'_{n+2})\dots(w,v'_{n'-1})(w,w')$$

is a dipath in $D \boxtimes D'$ connecting (w,v'_n) and (w,w').

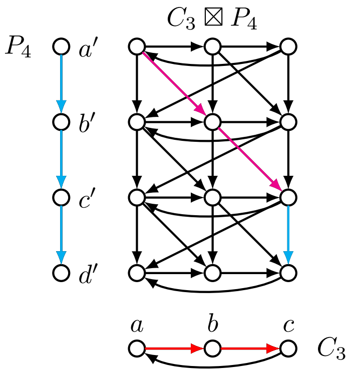
The two dipaths in $D \boxtimes D'$ intersect only at (w,v'_n). Thus, the concatenated sequence

$$(v,v')(v_1,v'_1)(v_2,v'_2)\dots(v_{n-1},v'_{n-1}) \\ (w,v_n)(w,v'_{n+1})(w,v'_{n+2})\dots(w,v'_{n'-1})(w,w')$$

is a dipath of length $n' \leq \max\{k,k'\}$ connecting (v,v') and (w,w').

The remaining cases when either $w = v$ or $w' = v'$ are analogous, with an exception of a small detail that either the (v,w)-dipath or the (v',w')-dipath is of length 0. $\square$

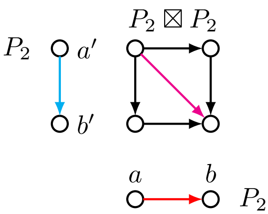
The procedure from the proof of Lemma 3.4 is illustrated below.



The red dipath abc in C_3 and the cyan dipath $a'b'c'$ in P_4 create a magenta dipath $(a,a')(b,b')(c,c')$ in $C_3 \boxtimes P_4$ and the dipath $c'd'$ in P_4 lifts to a dipath $(c,c')(c,d')$ in $C_3 \boxtimes P_4$. Concatenating the two paths in $C_3 \boxtimes P_4$ together yields the dipath $(a,a')(b,b')(c,c')(c,d')$ connecting (a,a') (c,d').

Lemma 3.5. *If (v,v') is an l-king in the strong product $D \boxtimes D'$ of digraphs D and D', then v is a k-king in D and v' is a k'-king in a digraph D' with $k \leq l$ and $k' \leq l$.*

Proof. The proof is almost identical to the proof Lemma 3.3 for the Cartesian product. However, in Lemma 3.3, we could further obtain $k + k' \leq l$. Here, we can only conclude that $k \leq l$ and $k' \leq l$ because the arcs in $D \boxtimes D'$ are not only "horizontal" and "vertical", but also "diagonal". This is illustrated on a simple example below when the magenta dipath $(a,a')(b,b')$ in $P_2 \boxtimes P_2$ projects into a red dipath ab in $D = P_2$ and a cyan dipath $a'b'$ in $D' = P_2$.



□

3.3 Kings in lexicographic products

The proof of the third equivalence in Theorem 1.1 will be established by the following two Lemmas.

Lemma 3.6. *If v is a k-king in a digraph D and either*

a) *v' is a k'-king in a digraph D', or*

b) *v lies on a dicycle of length k' in D,*

then (v, v') is a $\max\{k, k'\}$-king in the lexicographic product $D \circ D'$.

Proof. Pick any vertex (w, w') in $D \circ D'$. First, assume $v \neq w$. Since v is a k-king in D, there is an (v, w)-dipath $vv_1v_2 \dots v_{n-1}w$ of length $n \leq k$ in D. Thus, $(v, v')(v_1, w')(v_2, w') \dots (v_{n-1}, w')(w, w')$ is a dipath of length n in $D \circ D'$ connecting (v, v') and (w, w').

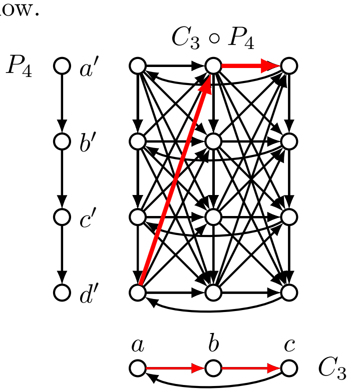
Second, assume $v = w$ and that v' is k'-king in D'. Thus, similarly as in the proof of Lemma 3.2, there is an (v', w')-dipath $v'v'_1v'_2 \dots v'_{n'-1}w'$ of length $n' \leq k'$ in D'. Thus, $(v, v')(v, v'_1)(v, v'_2) \dots (v, v'_{n'-1})(v, w')$ is a dipath in $D \circ D'$ connecting (v, v') and (w, w').

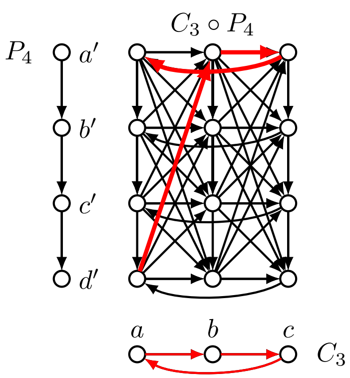
Finally, assume $v = w$ and that v lies on a dicycle of length k'. Any vertex of the dicycle can act as a starting vertex and thus v lies on a closed (v, v)-dipath $vv_1v_2 \dots v_{k'-1}v$ of length k'. Hence,

$$(v, v')(v_1, w')(v_2, w') \dots (v_{k'-1}, w')(v, w')$$

is a dipath of length k' in $D \circ D'$ connecting (v, v') and (w, w'). $\square$

The two crucial procedures from the proof of Lemma 3.6 are illustrated below.





The vertex (a, d') in the lexicographic product $C_3 \circ P_4$ is a 3-king despite the fact that d' is not a king in P_4. For example, we can connect (a, d') to (c, a') by a dipath of length 2 because a is a 2-king in C_3. And, we can connect (a, d) to (a, a') by a dipath of length 3 because a lies on a dicycle of length 3 in C_3.

Lemma 3.7. *If (v, v') is a k-king in the lexicographic product $D \circ D'$ of digraphs D and D', then v is a k-king in D, and either*

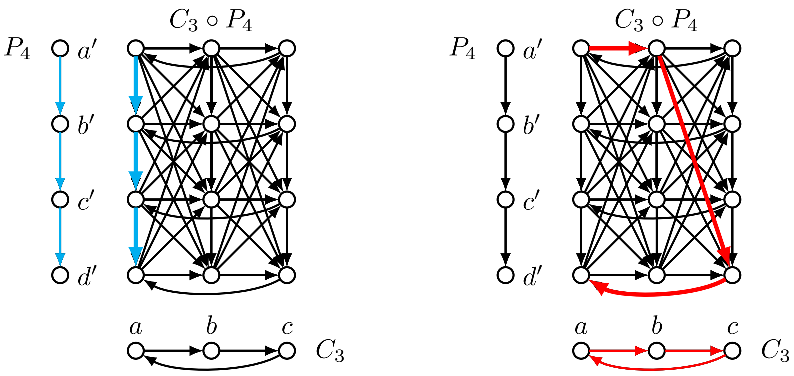
a) *v' is a k'-king in a digraph D' with $k' \leq k$, or*

b) *v lies on dicycle of length at most k.*

Proof. The proof that v is a k-king is analogous to the proof of the corresponding part in Lemma 3.3.

Now, let $w' \neq v'$ be any vertex in D' and $w = v$. Since (v, v') is an k-king, there exists a directed path $(v, v')(v_1, v_1')(v_2, v_2') \ldots (v_{n-1}, v_{n-1}')(v, w')$ in $D \circ D'$ of length $n \leq k$ connecting (v, v') and (v, w'). If $v = v_1 = v_2 = v_3 = \cdots = v_{n-1}$, then, as before, the vertex v' is a k-king in D'. Otherwise, the vertex v must lie on a dicycle of length at most k. $\square$

The proof of Lemma 3.7 is illustrated below.

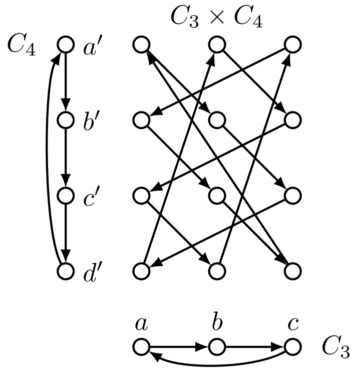


The vertex (a, a') in the lexicographic product $C_3 \circ P_4$ is a 3-king. It can reach the vertex (a, d') either by a dipath that is entirely in the fiber $\{a\} \times P_4$, or by a dipath that goes outside of the fiber $\{a\} \times P_4$. In the first case, the dipath projects naturally into an (a', d')-dipath in P_4 of length 3. In the second case, the dipath projects into a dicycle in C_3 of length 3 containing a.

3.4 Kings in direct products

The proof of the last implication of Theorem 1.1 is analogous to the proofs of Lemma 3.3 and Lemma 3.5.

While Theorem 1.2 gives the condition on when the implication can be reversed, unlike Theorem 1.1, Theorem 1.2 does not provide any quantitative information about the nature of the kings in the product. On an example of $C_3 \times C_4$ below, mentioned already in [13], we see that every vertex in C_3 is a 2-king, every vertex in C_4 is a 3-king, yet every vertex in $C_3 \times C_4$ is 11-king.



This seems to indicate that the only bound for the kings in the direct product $D \times D'$ is $k = \|D\| \cdot \|D'\| - 1$. However, it is not known how sharp this bound is.

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